C*-tensor categories and subfactors for totally disconnected groups

by Yuki Arano[1] and Stefaan Vaes[2]

In *Operator Algebras and Applications*, The Abel Symposium 2015, Springer, 2016, pp. 1-43.

Abstract

We associate a rigid C*-tensor category $\mathcal{C}$ to a totally disconnected locally compact group G and a compact open subgroup $K < G$. We characterize when $\mathcal{C}$ has the Haagerup property or property (T), and when $\mathcal{C}$ is weakly amenable. When G is compactly generated, we prove that $\mathcal{C}$ is essentially equivalent to the planar algebra associated by Jones and Burstein to a group acting on a locally finite bipartite graph. We then concretely realize $\mathcal{C}$ as the category of bimodules generated by a hyperfinite subfactor.

1 Introduction

Rigid C*-tensor categories arise as representation categories of compact groups and compact quantum groups and also as (part of) the standard invariant of a finite index subfactor. They can be viewed as a discrete group like structure and this analogy has lead to a lot of recent results with a flavor of geometric group theory, see [PV14, NY15a, GJ15, NY15b, PSV15].

In this paper, we introduce a rigid C*-tensor category $\mathcal{C}$ canonically associated with a totally disconnected locally compact group G and a compact open subgroup $K < G$. Up to Morita equivalence, $\mathcal{C}$ does not depend on the choice of K. The tensor category $\mathcal{C}$ can be described in several equivalent ways, see Section 2. Here, we mention that the representation category of K is a full subcategory of $\mathcal{C}$ and that the "quotient" of the fusion algebra of $\mathcal{C}$ by $\operatorname{Rep} K$ is the Hecke algebra of finitely supported functions on $K\backslash G/K$ equipped with the convolution product.

When G is compactly generated, we explain how the C*-tensor category $\mathcal{C}$ is related to the planar algebra $\mathcal{P}$ (i.e. standard invariant of a subfactor) associated in [J98, B10] with a locally finite bipartite graph $\mathcal{G}$ and a closed subgroup $G < \operatorname{Aut}(\mathcal{G})$. At the same time, we prove that these planar algebras $\mathcal{P}$ can be realized by a *hyperfinite* subfactor.

Given a finite index subfactor $N \subset M$, the notions of *amenability*, *Haagerup property* and *property (T)* for its standard invariant $\mathcal{G}_{N,M}$ were introduced by Popa in [P94a, P99, P01] in terms of the associated symmetric enveloping algebra $T \subset S$ (see [P94a, P99]) and shown to only depend on $\mathcal{G}_{N,M}$. Denoting by $\mathcal{C}$ the tensor category of M-M-bimodules generated by the subfactor, these properties were then formulated in [PV14] intrinsically in terms of $\mathcal{C}$, and in particular directly in terms of $\mathcal{G}_{N,M}$. We recall these definitions and equivalent formulations in Section 4. Similarly, *weak amenability* and the corresponding Cowling-Haagerup constant for the standard invariant $\mathcal{G}_{N,M}$ of a subfactor $N \subset M$ were first defined in terms of the symmetric enveloping inclusion in [Br14] and then intrinsically for rigid C*-tensor categories in [PV14],

[1]Graduate School of Mathematics, University of Tokyo, 3-8-1, Komaba, Meguro-ku, Tokyo (Japan), arano@ms.u-tokyo.ac.jp. Supported by the Research Fellow of the Japan Society for the Promotion of Science and the Program for Leading Graduate Schools, MEXT, Japan.

[2]KU Leuven, Department of Mathematics, Leuven (Belgium), stefaan.vaes@wis.kuleuven.be
Supported in part by European Research Council Consolidator Grant 614195, and by long term structural funding – Methusalem grant of the Flemish Government.

see Section 5. Reinterpreting [DFY13, A14], it was proved in [PV14] that the representation category of $\mathrm{SU}_q(2)$ (and thus, the Temperley-Lieb-Jones standard invariant) is weakly amenable and has the Haagerup property, while the representation category of $\mathrm{SU}_q(3)$ has property (T).

For the C*-tensor categories $\mathcal{C}$ that we associate to a totally disconnected group G, we characterize when $\mathcal{C}$ has the Haagerup property or property (T) and when $\mathcal{C}$ is weakly amenable. We give several examples and counterexamples, in particular illustrating that the Haagerup property/weak amenability of G is not sufficient for $\mathcal{C}$ to have the Haagerup property or to be weakly amenable. Even more so, when $\mathcal{C}$ is the category associated with $G = \mathrm{SL}(2, \mathbb{Q}_p)$, then the subcategory $\operatorname{Rep} K$ with $K = \mathrm{SL}(2, \mathbb{Z}_p)$ has the relative property (T). When $G = \mathrm{SL}(n, \mathbb{Q}_p)$ with $n \geq 3$, the tensor category $\mathcal{C}$ has property (T), but we also give examples of property (T) groups G such that $\mathcal{C}$ does not have property (T).

Our main technical tool is Ocneanu's tube algebra [O93] associated with any rigid C*-tensor category, see Section 3. When $\mathcal{C}$ is the C*-tensor category of a totally disconnected group G, we prove that the tube algebra is isomorphic with a canonical dense $*$-subalgebra of $C_0(G) \rtimes_{\mathrm{Ad}} G$, where G acts on G by conjugation. We can therefore express the above mentioned approximation and rigidity properties of the tensor category $\mathcal{C}$ in terms of G and the dynamics of the conjugation action $G \curvearrowright^{\mathrm{Ad}} G$.

In this paper, all locally compact groups are assumed to be second countable. We call totally disconnected group every second countable, locally compact, totally disconnected group.

2 C*-tensor categories of totally disconnected groups

Throughout this section, fix a totally disconnected group G. For all compact open subgroups $K_1, K_2 < G$, we define

$\mathcal{C}_1$: the category of K_1-K_2-$L^\infty(G)$-modules, i.e. Hilbert spaces $\mathcal{H}$ equipped with commuting unitary representations $(\lambda(k_1))_{k_1 \in K_1}$ and $(\rho(k_2))_{k_2 \in K_2}$ and with a normal $*$-representation $\Pi : L^\infty(G) \to B(\mathcal{H})$ that are equivariant with respect to the left translation action $K_1 \curvearrowright G$ and the right translation action $K_2 \curvearrowright G$;

$\mathcal{C}_2$: the category of K_1-$L^\infty(G/K_2)$-modules, i.e. Hilbert spaces $\mathcal{H}$ equipped with a unitary representation $(\pi(k_1))_{k_1 \in K_1}$ and a normal $*$-representation $\Pi : L^\infty(G/K_2) \to B(\mathcal{H})$ that are covariant with respect to the left translation action $K_1 \curvearrowright G/K_2$;

$\mathcal{C}_3$: the category of G-$L^\infty(G/K_1)$-$L^\infty(G/K_2)$-modules, i.e. Hilbert spaces $\mathcal{H}$ equipped with a unitary representation $(\pi(g))_{g \in G}$ and with an $L^\infty(G/K_1)$-$L^\infty(G/K_2)$-bimodule structure that are equivariant with respect to the left translation action of G on G/K_1 and G/K_2 ;

and with morphisms given by bounded operators that intertwine the given structure.

Let $K_3 < G$ also be a compact open subgroup. We define the tensor product $\mathcal{H} \otimes_{K_2} \mathcal{K}$ of a K_1-K_2-$L^\infty(G)$-module $\mathcal{H}$ and a K_2-K_3-$L^\infty(G)$-module $\mathcal{K}$ as the Hilbert space

$$\mathcal{H} \otimes_{K_2} \mathcal{K} = \{\xi \in \mathcal{H} \otimes \mathcal{K} \mid (\rho(k_2) \otimes \lambda(k_2))\xi = \xi \ \text{ for all } \ k_2 \in K_2\}$$

equipped with the unitary representations $(\lambda(k_1) \otimes 1)_{k_1 \in K_1}$ and $(1 \otimes \rho(k_3))_{k_3 \in K_3}$ and with the representation $(\Pi_{\mathcal{H}} \otimes \Pi_{\mathcal{K}}) \circ \Delta$ of $L^\infty(G)$, where $\Delta : L^\infty(G) \to L^\infty(G) \mathbin{\overline{\otimes}} L^\infty(G)$ is the comultiplication given by $(\Delta(F))(g, h) = F(gh)$ for all $g, h \in G$.

The tensor product of a G-$L^\infty(G/K_1)$-$L^\infty(G/K_2)$-module $\mathcal{H}$ and a G-$L^\infty(G/K_2)$-$L^\infty(G/K_3)$-module $\mathcal{K}$ is denoted as $\mathcal{H} \otimes_{L^\infty(G/K_2)} \mathcal{K}$ and defined as the Hilbert space

$$\begin{aligned}\mathcal{H} \otimes_{L^\infty(G/K_2)} \mathcal{K} &= \{\xi \in \mathcal{H} \otimes \mathcal{K} \mid \xi(1_{gK_2} \otimes 1) = (1 \otimes 1_{gK_2})\xi \;\text{ for all }\; gK_2 \in G/K_2\} \\ &= \bigoplus_{g \in G/K_2} \mathcal{H} \cdot 1_{gK_2} \otimes 1_{gK_2} \cdot \mathcal{K}\end{aligned}$$

with the unitary representation $(\pi_{\mathcal{H}}(g) \otimes \pi_{\mathcal{K}}(g))_{g \in G}$ and with the $L^\infty(G/K_1)$-$L^\infty(G/K_3)$-bimodule structure given by the left action of $1_{gK_1} \otimes 1$ for $gK_1 \in G/K_1$ and the right action of $1 \otimes 1_{hK_3}$ for $hK_3 \in G/K_3$.

We say that objects $\mathcal{H}$ are of *finite rank*

- $\mathcal{C}_1$: if $\mathcal{H}_{K_2} := \{\xi \in \mathcal{H} \mid \rho(k_2)\xi = \xi \;\text{ for all }\; k_2 \in K_2\}$ is finite dimensional ; as we will see in the proof of Proposition 2.2, this is equivalent with requiring that ${}_{K_1}\mathcal{H}$ is finite dimensional ;
- $\mathcal{C}_2$: if $\mathcal{H}$ is finite dimensional ;
- $\mathcal{C}_3$: if $1_{eK_1} \cdot \mathcal{H}$ is finite dimensional ; as we will see in the proof of Proposition 2.2, this is equivalent with requiring that $\mathcal{H} \cdot 1_{eK_2}$ is finite dimensional.

Altogether, we get that $\mathcal{C}_1$ and $\mathcal{C}_3$ are *C*-2-categories*. In both cases, the 0-cells are the compact open subgroups of G. For all compact open subgroups $K_1, K_2 < G$, the 1-cells are the categories $\mathcal{C}_i(K_1, K_2)$ defined above and $\mathcal{C}_i(K_1, K_2) \times \mathcal{C}_i(K_2, K_3) \to \mathcal{C}_i(K_1, K_3)$ is given by the tensor product operation that we just introduced. Restricting to finite rank objects, we get rigid C*-2-categories.

Another typical example of a C*-2-category is given by Hilbert bimodules over II_1 factors: the 0-cells are II_1 factors, the 1-cells are the categories $\mathrm{Bimod}_{M_1\text{-}M_2}$ of Hilbert M_1-M_2-bimodules and $\mathrm{Bimod}_{M_1\text{-}M_2} \times \mathrm{Bimod}_{M_2\text{-}M_3} \to \mathrm{Bimod}_{M_1\text{-}M_3}$ is given by the Connes tensor product. Again, restricting to finite index bimodules, we get a rigid C*-2-category.

Remark 2.1. The standard invariant of an extremal finite index subfactor $N \subset M$ can be viewed as follows as a rigid C*-2-category. There are only two 0-cells, namely N and M; the 1-cells are the N-N, N-M, M-N and M-M-bimodules generated by the subfactor; and we are given a favorite and generating 1-cell from N to M, namely the N-M-bimodule $L^2(M)$.

Abstractly, a rigid C*-2-category $\mathcal{C}$ with only two 0-cells (say $+$ and $-$), irreducible tensor units in $\mathcal{C}_{++}$ and $\mathcal{C}_{--}$, and a given generating object $\mathcal{H} \in \mathcal{C}_{+-}$ is exactly the same as a standard λ-lattice in the sense of Popa [P94b, Definitions 1.1 and 2.1]. Indeed, for every $n \geq 0$, define $\mathcal{H}_{+,n}$ as the n-fold alternating tensor product of $\mathcal{H}$ and $\overline{\mathcal{H}}$ starting with $\mathcal{H}$. Similarly, define $\mathcal{H}_{-,n}$ by starting with $\overline{\mathcal{H}}$. For $0 \leq j$, define $A_{0j} = \mathrm{End}(\mathcal{H}_{+,j})$. When $0 \leq i \leq j < \infty$, define $A_{ij} \subset A_{0j}$ as $A_{ij} := 1^i \otimes \mathrm{End}(\mathcal{H}_{(-1)^i, j-i})$ viewed as a subalgebra of $A_{0j} = \mathrm{End}(\mathcal{H}_{+,j})$ by writing $\mathcal{H}_{+,j} = \mathcal{H}_{+,i}\mathcal{H}_{(-1)^i, j-i}$. The standard solutions for the conjugate equations (see Section 3) give rise to canonical projections $e_+ \in \mathrm{End}(\mathcal{H}\overline{\mathcal{H}})$ and $e_- \in \mathrm{End}(\overline{\mathcal{H}}\mathcal{H})$ given by

$$e_+ = d(\mathcal{H})^{-1} s_{\mathcal{H}} s_{\mathcal{H}}^* \quad \text{and} \quad e_- = d(\mathcal{H})^{-1} t_{\mathcal{H}} t_{\mathcal{H}}^* \; ,$$

and thus to a representation of the Jones projections $e_j \in A_{kl}$ (for $k < j < l$). Finally, if we equip all A_{ij} with the normalized categorical trace, we have defined a standard λ-lattice in the sense of [P94b, Definitions 1.1 and 2.1]. Given two rigid C*-2-categories with fixed generating objects as above, it is straightforward to check that the associated standard λ-lattices are isomorphic if and only if there exists an equivalence of C*-2-categories preserving the generators. Conversely given a standard λ-lattice $\mathcal{G}$, by [P94b, Theorem 3.1], there exists an extremal subfactor $N \subset M$ whose standard invariant is $\mathcal{G}$ and we can define $\mathcal{C}$ as the C*-2-category of the subfactor $N \subset M$, generated by the N-M-bimodule $L^2(M)$ as in the beginning of this remark. One can also define $\mathcal{C}$ directly in terms of $\mathcal{G}$ (see e.g. [MPS08, Section 4.1] for a planar algebra version of this construction).

Thus, also subfactor planar algebras in the sense of [J99] are "the same" as rigid C*-2-categories with two 0-cells and such a given generating object $\mathcal{H} \in \mathcal{C}_{+-}$.

For more background on rigid C*-tensor categories, we refer to [NT13].

Proposition 2.2. *The C*-2-categories $\mathcal{C}_1$ and $\mathcal{C}_3$ are naturally equivalent. In particular, fixing $K_1 = K_2 = K$, we get the naturally equivalent rigid C*-tensor categories $\mathcal{C}_{1,f}(K < G)$ and $\mathcal{C}_{3,f}(K < G)$. Up to Morita equivalence[3], these do not depend on the choice of compact open subgroup $K < G$.*

Proof. Using the left and right translation operators λ_g and ρ_g on $L^2(G)$, one checks that the following formulae define natural equivalences and their inverses between the categories $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$.

- $\mathcal{C}_1 \to \mathcal{C}_2 : \mathcal{H} \mapsto \mathcal{H}_{K_2}$, where $\mathcal{H}_{K_2}$ is the space of right K_2-invariant vectors and where the K_1-$L^\infty(G/K_2)$-module structure on $\mathcal{H}_{K_2}$ is given by restricting the corresponding structure on $\mathcal{H}$.
- $\mathcal{C}_2 \to \mathcal{C}_1 : \mathcal{H} \mapsto \mathcal{H} \otimes_{L^\infty(G/K_2)} L^2(G)$ given by
$$\begin{aligned}\{\xi \in \mathcal{H} \otimes L^2(G) \mid (1_{gK_2} \otimes 1)\xi = (1 \otimes 1_{gK_2})\xi \;\text{ for all }\; g \in G\} \\ = \bigoplus_{g \in G/K_2} 1_{gK_2} \cdot \mathcal{H} \otimes L^2(gK_2)\end{aligned}$$
and where the K_1-K_2-$L^\infty(G)$-module structure is given by $(\lambda_{\mathcal{H}}(k_1) \otimes \lambda_{k_1})_{k_1 \in K_1}$, $(1 \otimes \rho_{k_2})_{k_2 \in K_2}$ and multiplication with $1 \otimes F$ when $F \in L^\infty(G)$.
- $\mathcal{C}_3 \to \mathcal{C}_2 : \mathcal{H} \mapsto 1_{eK_1} \cdot \mathcal{H}$ and where the K_1-$L^\infty(G/K_2)$-module structure on $1_{eK_1} \cdot \mathcal{H}$ is given by restricting the corresponding structure on $\mathcal{H}$.
- $\mathcal{C}_2 \to \mathcal{C}_3 : \mathcal{H} \mapsto L^2(G) \otimes_{K_1} \mathcal{H} = \{\xi \in L^2(G) \otimes \mathcal{H} \mid (\rho_{k_1} \otimes \pi(k_1))\xi = \xi \;\text{ for all }\; k_1 \in K_1\}$ and where the G-$L^\infty(G/K_1)$-$L^\infty(G/K_2)$-module structure is given by $(\lambda_g \otimes 1)_{g \in G}$, multiplication with $F \otimes 1$ for $F \in L^\infty(G/K_1)$ and multiplication with $(\mathrm{id} \otimes \Pi)\Delta(F)$ for $F \in L^\infty(G/K_2)$.

By definition, if $\mathcal{H} \in \mathcal{C}_1$ has finite rank, the Hilbert space $\mathcal{H}_{K_2}$ is finite dimensional. Conversely, if $\mathcal{K} \in \mathcal{C}_2$ and $\mathcal{K}$ is a finite dimensional Hilbert space, then the corresponding object $\mathcal{H} \in \mathcal{C}_1$ has the property that both ${}_{K_1}\mathcal{H}$ and $\mathcal{H}_{K_2}$ are finite dimensional. Therefore, $\mathcal{H} \in \mathcal{C}_1$ has finite rank if and only if ${}_{K_1}\mathcal{H}$ is a finite dimensional Hilbert space. A similar reasoning holds for objects in $\mathcal{C}_3$.

It is straightforward to check that the resulting equivalence $\mathcal{C}_1 \leftrightarrow \mathcal{C}_3$ preserves tensor products, so that we have indeed an equivalence between the C*-2-categories $\mathcal{C}_1$ and $\mathcal{C}_3$.

To prove the final statement in the proposition, it suffices to observe that for all compact open subgroups $K_1, K_2 < G$, we have that $L^2(K_1K_2)$ is a nonzero finite rank K_1-K_2-$L^\infty(G)$-module and that $L^2(G/(K_1 \cap K_2))$ is a nonzero finite rank G-$L^\infty(G/K_1)$-$L^\infty(G/K_2)$-module, so that $\mathcal{C}_{i,f}(K_1 < G)$ and $\mathcal{C}_{i,f}(K_2 < G)$ are Morita equivalent for $i = 1, 3$. □

[3] In the sense of [M01, Section 4], where the terminology weak Morita equivalence is used; see also [PSV15, Definition 7.3] and [NY15b, Section 3].

The rigid C*-2-categories $\mathcal{C}_1$ and $\mathcal{C}_2$ can as follows be fully faithfully embedded in the category of bimodules over the hyperfinite II_1 factor. We construct this embedding in an extremal way in the sense of subfactors (cf. Corollary 2.4).

To do so, given a totally disconnected group G, we fix a continuous action $G \curvearrowright^\alpha P$ of G on the hyperfinite II_∞ factor P that is *strictly outer* in the sense of [V03, Definition 2.1]: the relative commutant $P' \cap P \rtimes G$ equals $\mathbb{C}1$. Moreover, we should choose this action in such a way that $\mathrm{Tr} \circ \alpha_g = \Delta(g)^{-1/2} \mathrm{Tr}$ for all $g \in G$ (where Δ is the modular function on G) and such that there exists a projection $p \in P$ of finite trace with the property that $\alpha_k(p) = p$ whenever k belongs to a compact subgroup of G. Such an action indeed exists: write $P = R_0 \mathbin{\overline{\otimes}} R_1$ where R_0 is a copy of the hyperfinite II_1 factor and R_1 is a copy of the hyperfinite II_∞ factor. Choose a continuous trace scaling action $\mathbb{R}_0^+ \curvearrowright^{\alpha_1} R_1$. By [V03, Corollary 5.2], we can choose a strictly outer action $G \curvearrowright^{\alpha_0} R_0$. We then define $\alpha_g = (\alpha_0)_g \otimes (\alpha_1)_{\Delta(g)^{-1/2}}$ and we take $p = 1 \otimes p_1$, where $p_1 \in R_1$ is any projection of finite trace. Whenever k belongs to a compact subgroup of G, we have $\Delta(k) = 1$ and thus $\alpha_k(p) = p$.

Whenever $K_1, K_2 < G$ are compact open subgroups of G, we write

$$[K_1 : K_2] = [K_1 : K_1 \cap K_2] \, [K_2 : K_1 \cap K_2]^{-1} \; .$$

Fixing a left Haar measure λ on G, we have $[K_1 : K_2] = \lambda(K_1)\, \lambda(K_2)^{-1}$. Therefore, we have that $[K : gKg^{-1}] = \Delta(g)$ for all compact open subgroups $K < G$ and all $g \in G$.

Theorem 2.3. *Let G be a totally disconnected group and choose a strictly outer action $G \curvearrowright^\alpha P$ on the hyperfinite II_∞ factor P and a projection $p \in P$ as above. For every compact open subgroup $K < G$, write $R(K) = (pPp)^K$. Then each $R(K)$ is a copy of the hyperfinite II_1 factor.*

To every K_1-K_2-$L^\infty(G)$-module $\mathcal{H}$, we associate the Hilbert $R(K_1)$-$R(K_2)$-bimodule $\mathcal{K}$ given by (2.1) below. Then $\mathcal{H} \mapsto \mathcal{K}$ is a fully faithful 2-functor. Also, $\mathcal{H}$ has finite rank if and only if $\mathcal{K}$ is a finite index bimodule. In that case,

$$\dim_{R(K_1)-}(\mathcal{K}) = [K_1 : K_2]^{1/2} \, \dim_{\mathcal{C}_1}(\mathcal{H}) \quad \text{and} \quad \dim_{-R(K_2)}(\mathcal{K}) = [K_2 : K_1]^{1/2} \, \dim_{\mathcal{C}_1}(\mathcal{H}) \; ,$$

where $\dim_{\mathcal{C}_1}(\mathcal{H})$ is the categorical dimension of $\mathcal{H} \in \mathcal{C}_1$.

Proof. Given a K_1-K_2-$L^\infty(G)$-module $\mathcal{H}$, turn $\mathcal{H} \otimes L^2(P)$ into a Hilbert $(P \rtimes K_1)$-$(P \rtimes K_2)$-bimodule via

$$\begin{aligned} &u_k \cdot (\xi \otimes b) \cdot u_r = \lambda(k)\rho(r)^* \xi \otimes \alpha_r^{-1}(b) && \text{for all } \; k \in K_1, r \in K_2, \xi \in \mathcal{H}, b \in L^2(P) \; , \\ &a \cdot \zeta \cdot d = (\Pi \otimes \mathrm{id})\alpha(a) \, \zeta \, (1 \otimes d) && \text{for all } \; a, d \in P, \zeta \in \mathcal{H} \otimes L^2(P) \; , \end{aligned}$$

where $\alpha : P \to L^\infty(G) \mathbin{\overline{\otimes}} P$ is given by $(\alpha(a))(g) = \alpha_g^{-1}(a)$.

Whenever $K < G$ is a compact open subgroup, we define the projection $p_K \in L(G)$ given by

$$p_K = \lambda(K)^{-1} \int_K \lambda_k \, dk \; .$$

We also write $e_K = pp_K$ viewed as a projection in $P \rtimes K$. Since $P \subset P \rtimes K \subset P \rtimes G$, we have that $P' \cap (P \rtimes K) = \mathbb{C}1$, so that $P \rtimes K$ is a factor. So, $P \rtimes K$ is a copy of the hyperfinite II_∞ factor and $e_K \in P \rtimes K$ is a projection of finite trace. We identify $R(K) = e_K(P \rtimes K)e_K$ through the bijective $*$-isomorphism $(pPp)^K \to e_K(P \rtimes K)e_K : a \mapsto ap_K$. In particular, $R(K)$ is a copy of the hyperfinite II_1 factor.

So, for every K_1-K_2-$L^\infty(G)$-module $\mathcal{H}$, we can define the $R(K_1)$-$R(K_2)$-bimodule

$$\mathcal{K} = e_{K_1} \cdot (\mathcal{H} \otimes L^2(P)) \cdot e_{K_2} \; . \tag{2.1}$$

We claim that $\mathrm{End}_{R(K_1)-R(K_2)}(\mathcal{K}) = \mathrm{End}_{\mathcal{C}_1}(\mathcal{H})$ naturally. More concretely, we have to prove that

$$\mathrm{End}_{(P \rtimes K_1)-(P \rtimes K_2)}(\mathcal{H} \otimes L^2(P)) = \mathrm{End}_{\mathcal{C}_1}(\mathcal{H}) \otimes 1 \; , \tag{2.2}$$

where $\mathrm{End}_{\mathcal{C}_1}(\mathcal{H})$ consists of all bounded operators on $\mathcal{H}$ that commute with $\lambda(K_1)$, $\rho(K_2)$ and $\Pi(L^\infty(G))$. To prove (2.2), it is sufficient to show that

$$\mathrm{End}_{P-P}(\mathcal{H} \otimes L^2(P)) = \Pi(L^\infty(G))' \otimes 1 \; . \tag{2.3}$$

Note that the left hand side of (2.3) equals $(\Pi \otimes \mathrm{id})\alpha(P)' \cap B(\mathcal{H}) \mathbin{\overline{\otimes}} P$. Assume that $T \in (\Pi \otimes \mathrm{id})\alpha(P)' \cap B(\mathcal{H}) \mathbin{\overline{\otimes}} P$. In the same was as in [V03, Proposition 2.7], it follows that $T \in \Pi(L^\infty(G))' \cap 1$. For completeness, we provide a detailed argument. Define the unitary $W \in L^\infty(G) \mathbin{\overline{\otimes}} L(G)$ given by $W(g) = \lambda_g$. We view both T and $(\Pi \otimes \mathrm{id})(W)$ as elements in $B(\mathcal{H}) \mathbin{\overline{\otimes}} (P \rtimes G)$. For all $a \in P$, we have

$$\begin{aligned}(\Pi \otimes \mathrm{id})(W)\, T\, (\Pi \otimes \mathrm{id})(W)^*\, (1 \otimes a) &= (\Pi \otimes \mathrm{id})(W)\, T\, (\Pi \otimes \mathrm{id})\alpha(a)\, (\Pi \otimes \mathrm{id})(W)^* \\ &= (1 \otimes a)\, (\Pi \otimes \mathrm{id})(W)\, T\, (\Pi \otimes \mathrm{id})(W)^* \; .\end{aligned}$$

Since the action α is strictly outer, we conclude that $(\Pi \otimes \mathrm{id})(W)\, T\, (\Pi \otimes \mathrm{id})(W)^* = S \otimes 1$ for some $S \in B(\mathcal{H})$. So,

$$T = (\Pi \otimes \mathrm{id})(W)^*\, (S \otimes 1)\, (\Pi \otimes \mathrm{id})(W) \; .$$

The left hand side belongs to $B(\mathcal{H}) \mathbin{\overline{\otimes}} P$, while the right hand side belongs to $B(\mathcal{H}) \otimes L(G)$, and both are viewed inside $B(\mathcal{H}) \mathbin{\overline{\otimes}} (P \rtimes G)$. Since $P \cap L(G) = \mathbb{C}1$, we conclude that $T = T_0 \otimes 1$ for some $T_0 \in B(\mathcal{H})$ and that

$$T_0 \otimes 1 = (\Pi \otimes \mathrm{id})(W)^*\, (S \otimes 1)\, (\Pi \otimes \mathrm{id})(W) \; .$$

Defining the normal $*$-homomorphism $\Psi : L(G) \to L(G) \mathbin{\overline{\otimes}} L(G)$ given by $\Psi(\lambda_g) = \lambda_g \otimes \lambda_g$ for all $g \in G$, we apply $\mathrm{id} \otimes \Psi$ and conclude that

$$\begin{aligned}T_0 \otimes 1 \otimes 1 &= (\Pi \otimes \mathrm{id})(W)^*_{13}\, (\Pi \otimes \mathrm{id})(W)^*_{12}\, (S \otimes 1)\, (\Pi \otimes \mathrm{id})(W)_{12}\, (\Pi \otimes \mathrm{id})(W)_{13} \\ &= (\Pi \otimes \mathrm{id})(W)^*_{13}\, (T_0 \otimes 1 \otimes 1)\, (\Pi \otimes \mathrm{id})(W)_{13} \; .\end{aligned}$$

It follows that T_0 commutes with $\Pi(L^\infty(G))$ and (2.2) is proven.

It is easy to check that $\mathcal{H} \mapsto \mathcal{K}$ naturally preserves tensor products. So, we have found a fully faithful 2-functor from $\mathcal{C}_1$ to the C*-2-category of Hilbert bimodules over hyperfinite II$_1$ factors.

To compute $\dim_{-R(K_2)}(\mathcal{K})$, observe that for all $k \in K_1$, $r \in K_2$ and $g \in G$, we have $\alpha_{kgr}(p) = \alpha_{kg}(p) = \alpha_g(\alpha_{g^{-1}kg}(p)) = \alpha_g(p)$. Therefore, as a right $(P \rtimes K_2)$-module, we have

$$e_{K_1} \cdot (\mathcal{H} \otimes L^2(P)) \cong \bigoplus_{g \in K_1 \backslash G / K_2} \big(\mathcal{L}_g \otimes L^2(p_g P)\big) \; ,$$

where $p_g = \alpha_g^{-1}(p)$, where the Hilbert space $\mathcal{L}_g := \Pi(1_{K_1 g K_2})(\,_{K_1}\mathcal{H})$ comes with the unitary representation $(\rho(r))_{r \in K_2}$ and where the right $(P \rtimes K_2)$-module structure on $\mathcal{L}_g \otimes L^2(p_g P)$ is given by

$$(\xi \otimes b) \cdot (d u_r) = \rho(r)^* \xi \otimes \alpha_r^{-1}(bd) \quad \text{for all} \;\; \xi \in \mathcal{L}_g, b \in L^2(p_g P), d \in P, r \in K_2 \; .$$

Since $p_g P p_g \rtimes K_2 = p_g(P \rtimes K_2)p_g$ is a factor (actually, $K_2 \curvearrowright p_g P p_g$ is a so-called minimal action), it follows from [W88, Theorem 12] that there exists a unitary $V_g \in B(\mathcal{L}_g) \mathbin{\overline{\otimes}} p_g P p_g$ satisfying

$$(\mathrm{id} \otimes \alpha_r)(V_g) = V_g(\rho(r) \otimes 1) \quad \text{for all } \; r \in K_2 \; .$$

Then left multiplication with V_g intertwines the right $(P \rtimes K_2)$-module structure on the Hilbert space $\mathcal{L}_g \otimes L^2(p_g P)$ with the right $(P \rtimes K_2)$-module structure given by

$$(\xi \otimes b) \cdot (d u_r) = \xi \otimes \alpha_r^{-1}(bd) \quad \text{for all } \; \xi \in \mathcal{L}_g, b \in L^2(p_g P), d \in P, r \in K_2 \; .$$

Therefore,

$$\begin{aligned} \dim_{-R(K_2)}\big(\mathcal{L}_g \otimes L^2(p_g P)\big) \cdot e_{K_2} &= \dim(\mathcal{L}_g) \, \dim_{-(pPp)^{K_2}}\big(L^2(p_g P^{K_2} p)\big) \\ &= \dim(\mathcal{L}_g) \, \frac{\mathrm{Tr}(p_g)}{\mathrm{Tr}(p)} = \dim(\mathcal{L}_g) \, \Delta(g)^{1/2} \; . \end{aligned}$$

So, we have proved that

$$\dim_{-R(K_2)}(\mathcal{K}) = \sum_{g \in K_1 \backslash G / K_2} \dim\big(\Pi(1_{K_1 g K_2})(\; {}_{K_1}\mathcal{H})\big) \; \Delta(g)^{1/2} \; .$$

We similarly get that

$$\dim_{R(K_1)-}(\mathcal{K}) = \sum_{g \in K_1 \backslash G / K_2} \dim\big(\Pi(1_{K_1 g K_2})(\mathcal{H}_{K_2})\big) \; \Delta(g)^{-1/2} \; .$$

To make the connection with the categorical dimension of $\mathcal{H}$, it is useful to view $\mathcal{H}$ as the image of a G-$L^\infty(G/K_1)$-$L^\infty(G/K_2)$-module $\mathcal{H}'$ under the equivalence of Proposition 2.2. This means that we can view $\mathcal{H}$ as the space of L^2-functions $\xi : G \to \mathcal{H}'$ with the property that $\xi(g) \in 1_{eK_1} \cdot \mathcal{H}' \cdot 1_{gK_2}$ for a.e. $g \in G$. The $L^\infty(G)$-module structure of $\mathcal{H}$ is given by pointwise multiplication, while the K_1-K_2-module structure on $\mathcal{H}$ is given by

$$(k \cdot \xi \cdot r)(g) = \pi(k)\xi(k^{-1} g r^{-1}) \quad \text{for all } \; k \in K_1, r \in K_2, g \in G \; .$$

With this picture, it is easy to see that

$$\Pi(1_{K_1 g K_2})(\mathcal{H}_{K_2}) \cong 1_{eK_1} \cdot \mathcal{H}' \cdot 1_{K_1 g K_2} \; .$$

The map $\xi \mapsto \widetilde{\xi}$ with $\widetilde{\xi}(g) = \pi(g)^* \xi(g)$ is an isomorphism between $\mathcal{H}$ and the space of L^2-functions $\eta : G \to \mathcal{H}'$ with the property that $\eta(g) \in 1_{g^{-1}K_1} \cdot \mathcal{H}' \cdot 1_{eK_2}$ for a.e. $g \in G$. The $L^\infty(G)$-module structure is still given by pointwise multiplication, while the K_1-K_2-module structure is now given by

$$(k \cdot \eta \cdot r)(g) = \pi(r)^* \eta(k^{-1} g r^{-1}) \; .$$

In this way, we get that

$$\Pi(1_{K_1 g K_2})(\; {}_{K_1}\mathcal{H}) \cong 1_{K_2 g^{-1} K_1} \cdot \mathcal{H}' \cdot 1_{eK_2} \; .$$

It thus follows that

$$\dim_{-R(K_2)}(\mathcal{K}) = \sum_{g \in K_1 \backslash G / K_2} \dim(1_{K_2 g^{-1} K_1} \cdot \mathcal{H}' \cdot 1_{eK_2}) \, \Delta(g)^{1/2} \quad \text{and} \tag{2.4}$$

$$\dim_{R(K_1)-}(\mathcal{K}) = \sum_{g \in K_1 \backslash G / K_2} \dim(1_{eK_1} \cdot \mathcal{H}' \cdot 1_{K_1 g K_2}) \, \Delta(g)^{-1/2} \; . \tag{2.5}$$

Also note that for every $g \in G$, we have

$$\begin{aligned}\dim(1_{K_2 g^{-1} K_1} \cdot \mathcal{H}' \cdot 1_{eK_2}) &= [K_2 : K_2 \cap g^{-1} K_1 g] \; \dim(1_{g^{-1} K_1} \cdot \mathcal{H}' \cdot 1_{eK_2}) \\ &= [K_2 : K_2 \cap g^{-1} K_1 g] \; \dim(1_{eK_1} \cdot \mathcal{H}' \cdot 1_{gK_2}) \\ &= \frac{[K_2 : K_2 \cap g^{-1} K_1 g]}{[K_1 : K_1 \cap g K_2 g^{-1}]} \; \dim(1_{eK_1} \cdot \mathcal{H}' \cdot 1_{K_1 g K_2}) \\ &= [K_2 : K_1] \, \Delta(g)^{-1} \; \dim(1_{eK_1} \cdot \mathcal{H}' \cdot 1_{K_1 g K_2}) \; .\end{aligned}$$

It follows that

$$\begin{aligned}\dim_{-R(K_2)}(\mathcal{K}) &= [K_2 : K_1] \sum_{g \in K_1 \backslash G / K_2} \dim(1_{eK_1} \cdot \mathcal{H}' \cdot 1_{K_1 g K_2}) \, \Delta(g)^{-1/2} \\ &= [K_2 : K_1] \; \dim_{R(K_1)-}(\mathcal{K}) \; .\end{aligned}$$

If $\mathcal{H}$ has finite rank, also $\mathcal{H}'$ has finite rank so that $\mathcal{H}' \cdot 1_{eK_2}$ and $1_{eK_1} \cdot \mathcal{H}'$ are finite dimensional Hilbert spaces. It then follows that $\mathcal{K}$ is a finite index bimodule.

Conversely, assume that $\mathcal{K}$ has finite index. For every $g \in G$, write

$$\kappa(g) := \dim(1_{K_2 g^{-1} K_1} \cdot \mathcal{H}' \cdot 1_{eK_2}) \, \Delta(g)^{1/2} = [K_2 : K_1] \; \dim(1_{eK_1} \cdot \mathcal{H}' \cdot 1_{K_1 g K_2}) \, \Delta(g)^{-1/2} \; .$$

So,

$$\kappa(g)^2 = [K_2 : K_1] \, \dim(1_{K_2 g^{-1} K_1} \cdot \mathcal{H}' \cdot 1_{eK_2}) \, \dim(1_{eK_1} \cdot \mathcal{H}' \cdot 1_{K_1 g K_2}) \; .$$

Thus, whenever $\kappa(g) \neq 0$, we have that $\kappa(g) \geq [K_2 : K_1]^{1/2}$. Since

$$\dim_{-R(K_2)}(\mathcal{K}) = \sum_{g \in K_1 \backslash G / K_2} \kappa(g) \; ,$$

we conclude that there are only finitely many $g \in K_1 \backslash G / K_2$ for which $1_{K_2 g^{-1} K_1} \cdot \mathcal{H}' \cdot 1_{eK_2}$ is nonzero and for each of them, it is a finite dimensional Hilbert space. This implies that $\mathcal{H}' \cdot 1_{eK_2}$ is finite dimensional, so that $\mathcal{H}'$ has finite rank.

We have proved that $\mathcal{H} \mapsto \mathcal{K}$ is a fully faithful 2-functor from $\mathcal{C}_{1,f}$ to the finite index bimodules over hyperfinite II_1 factors. Moreover, for given compact open subgroups $K_1, K_2 < G$, the ratio between $\dim_{R(K_1)-}(\mathcal{K})$ and $\dim_{-R(K_2)}(\mathcal{K})$ equals $[K_1 : K_2]$ for all finite rank K_1-K_2-$L^\infty(G)$-modules $\mathcal{H}$. Since the functor is fully faithful, this then also holds for all $R(K_1)$-$R(K_2)$-subbimodules of $\mathcal{K}$. It follows that the categorical dimension of $\mathcal{K}$ equals

$$[K_2 : K_1]^{1/2} \, \dim_{R(K_1)-}(\mathcal{K}) = [K_1 : K_2]^{1/2} \, \dim_{-R(K_2)}(\mathcal{K}) \; .$$

Since the functor is fully faithful, the categorical dimensions of $\mathcal{H} \in \mathcal{C}_{1,f}$ and $\mathcal{K} \in \mathrm{Bimod}_f$ coincide, so that

$$[K_2 : K_1]^{1/2} \, \dim_{R(K_1)-}(\mathcal{K}) = \dim_{\mathcal{C}_1}(\mathcal{H}) = [K_1 : K_2]^{1/2} \, \dim_{-R(K_2)}(\mathcal{K}) \; . \tag{2.6}$$

□

Corollary 2.4. *Let G be a totally disconnected group with compact open subgroups $K_\pm < G$ and assume that $\mathcal{H}$ is a finite rank G-$L^\infty(G/K_+)$-$L^\infty(G/K_-)$-module. Denote by $\mathcal{C} = (\mathcal{C}_{++}, \mathcal{C}_{+-}, \mathcal{C}_{-+}, \mathcal{C}_{--})$ the C*-2-category of G-$L^\infty(G/K_\pm)$-$L^\infty(G/K_\pm)$-modules (with 0-cells K_+ and K_-) generated by the alternating tensor products of $\mathcal{H}$ and its adjoint.*

Combining Proposition 2.2 and Theorem 2.3, we find an extremal hyperfinite subfactor $N \subset M$ whose standard invariant, viewed as the C-2-category of N-N, N-M, M-N and M-M-bimodules generated by the N-M-bimodule $L^2(M)$, is equivalent with $(\mathcal{C}, \mathcal{H})$ (cf. Remark 2.1).*

Proof. A combination of Proposition 2.2 and Theorem 2.3 provides the finite index $R(K_+)$-$R(K_-)$-bimodule $\mathcal{K}$ associated with $\mathcal{H}$. Take nonzero projections $p_\pm \in R(K_\pm)$ such that writing $N = p_+ R(K_+) p_+$ and $M = p_- R(K_-) p_-$, we have that $\dim_{-M}(p_+ \cdot \mathcal{K} \cdot p_-) = 1$. We can then view $N \subset M$ in such a way that $L^2(M) \cong p_+ \cdot \mathcal{K} \cdot p_-$ as N-M-bimodules. The C*-2-category of N-N, N-M, M-N and M-M-bimodules generated by the N-M-bimodule $L^2(M)$ is by construction equivalent with the rigid C*-2-category of $R(K_\pm)$-$R(K_\pm)$-bimodules generated by $\mathcal{K}$. Since the 2-functor in Theorem 2.3 is fully faithful, this C*-2-category is equivalent with $\mathcal{C}$ and this equivalence maps the N-M-bimodule $L^2(M)$ to $\mathcal{H} \in \mathcal{C}_{+-}$. □

From Corollary 2.4, we get the following result.

Proposition 2.5. *Let $\mathcal{P}$ be the subfactor planar algebra of [J98, B10] associated with a connected locally finite bipartite graph $\mathcal{G}$, with edge set $\mathcal{E}$ and source and target maps $s : \mathcal{E} \to V_+$, $t : \mathcal{E} \to V_-$, together with[4] a closed subgroup $G < \mathrm{Aut}(\mathcal{G})$ acting transitively on V_+ as well as on V_-. Fix vertices $v_\pm \in V_\pm$ and write $K_\pm = \mathrm{Stab}\, v_\pm$.*

There exists an extremal hyperfinite subfactor $N \subset M$ whose standard invariant is isomorphic with $\mathcal{P}$. We have $[M : N] = \delta^2$ where

$$\begin{aligned}\delta &= \sum_{w \in V_-} \#\{e \in \mathcal{E} \mid s(e) = v_+, t(e) = w\}\, [\mathrm{Stab}\, w : \mathrm{Stab}\, v_+]^{1/2} \\ &= \sum_{w \in V_+} \#\{e \in \mathcal{E} \mid s(e) = w, t(e) = v_-\}\, [\mathrm{Stab}\, w : \mathrm{Stab}\, v_-]^{1/2} .\end{aligned}$$

Moreover, $\mathcal{P}$ can be described as the rigid C-2-category $\mathcal{C}_{3,f}(G, K_\pm, K_\pm)$ of all finite rank G-$L^\infty(G/K_\pm)$-$L^\infty(G/K_\pm)$-modules together with the generating object $\ell^2(\mathcal{E}) \in \mathcal{C}_{3,f}(G, K_+, K_-)$ (cf. Remark 2.1).*

Proof. We are given $G \curvearrowright \mathcal{E}$ and $G \curvearrowright V_+$, $G \curvearrowright V_-$ such that the source and target maps s, t are G-equivariant and such that G acts transitively on V_+ and on V_-. Put $K_\pm = \mathrm{Stab}\, v_\pm$ and note that $K_\pm < G$ are compact open subgroups. We identify $G/K_\pm = V_\pm$ via the map $gK_\pm \mapsto g \cdot v_\pm$. In this way, $\mathcal{H} := \ell^2(\mathcal{E})$ naturally becomes a finite rank G-$L^\infty(G/K_+)$-$L^\infty(G/K_-)$-module. Denote by $\mathcal{C}$ the C*-2-category of G-$L^\infty(G/K_\pm)$-$L^\infty(G/K_\pm)$-modules generated by the alternating tensor products of $\mathcal{H}$ and its adjoint.

In the 2-category $\mathcal{C}_3$, the n-fold tensor product $\mathcal{H} \otimes \overline{\mathcal{H}} \otimes \cdots$ equals $\ell^2(\mathcal{E}_{+,n})$, where $\mathcal{E}_{+,n}$ is the set of paths in the graph $\mathcal{G}$ starting at an even vertex and having length n. Similarly, the n-fold tensor product $\overline{\mathcal{H}} \otimes \mathcal{H} \otimes \cdots$ equals $\ell^2(\mathcal{E}_{-,n})$, where $\mathcal{E}_{-,n}$ is the set of paths of length n starting at an odd vertex. So by construction, under the equivalence of Remark 2.1, $\mathcal{C}$ together with its generator $\mathcal{H} \in \mathcal{C}_{+-}$ corresponds exactly to the planar algebra $\mathcal{P}$ constructed in [B10, J98].

By Corollary 2.4, we get that $(\mathcal{C}, \mathcal{H})$ is the standard invariant of an extremal hyperfinite subfactor $N \subset M$. In particular, $[M : N] = \delta^2$ with $\delta = \dim_{\mathcal{C}_3}(\mathcal{H})$. Combining (2.6) with (2.4), and using that

$$\Delta(g)^{-1/2} = [gK_+g^{-1} : K_+]^{1/2} = [\mathrm{Stab}(g \cdot v_+) : K_+]^{1/2} ,$$

[4]Note that in [B10], also a weight function $\mu : V_+ \sqcup V_- \to \mathbb{R}_0^+$ scaled by the action of G is part of the construction. But only when we take μ to be a multiple of the function $v \mapsto [\mathrm{Stab}\, v : \mathrm{Stab}\, v_+]^{1/2}$, we actually obtain a subfactor planar algebra, contrary to what is claimed in [B10, Proposition 4.1].

we get that

$$\begin{aligned}\delta &= [K_+ : K_-]^{1/2} \sum_{g \in G/K_+} \dim(1_{gK_+} \cdot \mathcal{H} \cdot 1_{eK_-})\, \Delta(g)^{-1/2} \\ &= \sum_{g \in G/K_+} \#\{e \in \mathcal{E} \mid s(e) = g \cdot v_+, t(e) = v_-\}\, [\mathrm{Stab}(g \cdot v_+) : K_+]^{1/2}\, [K_+ : K_-]^{1/2} \\ &= \sum_{w \in V_+} \#\{e \in \mathcal{E} \mid s(e) = w, t(e) = v_-\}\, [\mathrm{Stab}\, w : \mathrm{Stab}\, v_-]^{1/2} \; .\end{aligned}$$

Combining (2.6) with (2.5), we similarly get that

$$\delta = \sum_{w \in V_-} \#\{e \in \mathcal{E} \mid s(e) = v_+, t(e) = w\}\, [\mathrm{Stab}\, w : \mathrm{Stab}\, v_+]^{1/2} \; .$$

To conclude the proof of the proposition, it remains to show that $\mathcal{C}$ is equal to the C*-2-category of all finite rank G-$L^\infty(G/K_\pm)$-$L^\infty(G/K_\pm)$-modules. For the G-$L^\infty(G/K_+)$-$L^\infty(G/K_-)$-modules, this amounts to proving that all irreducible representations of $K_+ \cap K_-$ appear in

$$\ell^2(\text{paths starting at } v_+ \text{ and ending at } v_-) \; .$$

Since the graph is connected, the action of $K_+ \cap K_-$ on this set of paths is faithful and the result follows. The other cases are proved in the same way. □

Remark 2.6. Note that the subfactors $N \subset M$ in Proposition 2.5 are *irreducible* precisely when G acts transitively on the set of edges and there are no multiple edges. This means that the totally disconnected group G is *generated* by the compact open subgroups $K_\pm < G$ and that we can identify $\mathcal{E} = G/(K_+ \cap K_-)$, $V_\pm = G/K_\pm$ with the natural source and target maps $G/(K_+ \cap K_-) \to G/K_\pm$. The irreducible subfactor $N \subset M$ then has integer index given by $[M : N] = [K_+ : K_+ \cap K_-]\,[K_- : K_+ \cap K_-]$.

We finally note that the rigid C*-tensor categories $\mathcal{C}_{1,f}(K < G)$ and $\mathcal{C}_{3,f}(K < G)$ also arise in a different way as categories of bimodules over a II$_1$ factor in the case where $K < G$ is the *Schlichting completion* of a *Hecke pair* $\Lambda < \Gamma$, cf. [DV10, Section 4].

Recall that a Hecke pair consists of a countable group Γ together with a subgroup $\Lambda < \Gamma$ that is almost normal, meaning that $g\Lambda g^{-1} \cap \Lambda$ has finite index in Λ for all $g \in \Gamma$. The left translation action of Γ on Γ/Λ gives a homomorphism π of Γ to the group of permutations of Γ/Λ. The closure G of $\pi(\Gamma)$ for the topology of pointwise convergence is a totally disconnected group and the stabilizer K of the point $e\Lambda \in \Gamma/\Lambda$ is a compact open subgroup of G with the property that $\Lambda = \pi^{-1}(K)$. One calls (G, K) the Schlichting completion of the Hecke pair (Γ, Λ). Note that there is a natural identification of G/K and Γ/Λ.

Proposition 2.7. *Let $\Lambda < \Gamma$ be a Hecke pair with Schlichting completion $K < G$. Choose an action $\Gamma \curvearrowright^\alpha P$ of Γ by outer automorphisms of a II$_1$ factor P. Define $N = P \rtimes \Lambda$ and $M = P \rtimes \Gamma$. Note that $N \subset M$ is an irreducible, quasi-regular inclusion of II$_1$ factors. Denote by $\mathcal{C}$ the tensor category of finite index N-N-bimodules generated by the finite index N-subbimodules of $L^2(M)$.*

Then, $\mathcal{C}$ and the earlier defined $\mathcal{C}_{1,f}(K < G)$ and $\mathcal{C}_{3,f}(K < G)$ are naturally equivalent rigid C-tensor categories.*

Proof. Define

$\mathcal{C}_4$: the category of Λ-Λ-$\ell^\infty(\Gamma)$-modules, i.e. Hilbert spaces $\mathcal{H}$ equipped with two commuting unitary representations of Λ and a representation of $\ell^\infty(\Gamma)$ that are covariant with respect to the left and right translation actions $\Lambda \curvearrowright \Gamma$;

$\mathcal{C}_5$: the category of Λ-$\ell^\infty(\Gamma/\Lambda)$-modules, i.e. Hilbert spaces equipped with a unitary representation of Λ and a representation of $\ell^\infty(\Gamma/\Lambda)$ that are covariant with respect to the left translation action $\Lambda \curvearrowright \Gamma/\Lambda$:

with morphisms again given by bounded operators that intertwine the given structure.

To define the tensor product of two objects in $\mathcal{C}_4$, it is useful to view $\mathcal{H} \in \mathcal{C}_4$ as a family of Hilbert spaces $(\mathcal{H}_g)_{g\in\Gamma}$ together with unitary operators $\lambda(k) : \mathcal{H}_g \to \mathcal{H}_{kg}$ and $\rho(k) : \mathcal{H}_g \to \mathcal{H}_{gk^{-1}}$ for all $k \in \Lambda$, satisfying the obvious relations. The tensor product of two Λ-Λ-$\ell^\infty(\Gamma)$-modules $\mathcal{H}$ and $\mathcal{K}$ is then defined as

$$(\mathcal{H} \otimes_\Lambda \mathcal{K})_g = \Big\{ (\xi_h)_{h\in\Gamma} \;\Big|\; \begin{aligned} &\xi_h \in \mathcal{H}_h \otimes \mathcal{K}_{h^{-1}g} \;, \\ &\xi_{hk^{-1}} = (\rho_{\mathcal{H}}(k) \otimes \lambda_{\mathcal{K}}(k))(\xi_h) \text{ for all } h \in \Gamma, k \in \Lambda \;, \\ &\sum_{h\in\Gamma/\Lambda} \|\xi_h\|^2 < \infty \end{aligned} \Big\}$$

with $\lambda(k) : (\mathcal{H} \otimes_\Lambda \mathcal{K})_g \to (\mathcal{H} \otimes_\Lambda \mathcal{K})_{kg}$ given by $(\lambda(k)\xi)_h = (\lambda_{\mathcal{H}}(k) \otimes 1)\xi_{k^{-1}h}$ and $\rho(k) : (\mathcal{H} \otimes_\Lambda \mathcal{K})_g \to (\mathcal{H} \otimes_\Lambda \mathcal{K})_{gk^{-1}}$ given by $(\rho(k)\xi)_h = (1 \otimes \rho_{\mathcal{K}}(k))\xi(h)$ for all $k \in \Lambda$, $h \in \Gamma$. Of course, choosing a section $i : \Gamma/\Lambda \to \Gamma$, we have

$$(\mathcal{H} \otimes_\Lambda \mathcal{K})_g \cong \bigoplus_{h\in\Gamma/\Lambda} (\mathcal{H}_{i(h)} \otimes \mathcal{K}_{i(h)^{-1}g}) \;,$$

but this isomorphism depends on the choice of the section.

As in Proposition 2.2, $\mathcal{C}_4$ and $\mathcal{C}_5$ are equivalent C*-categories, where the equivalence and its inverse are defined as follows.

- $\mathcal{C}_4 \to \mathcal{C}_5 : \mathcal{H} \mapsto \mathcal{K}$, with

$$\mathcal{K}_{g\Lambda} = \big\{ (\xi_h)_{h\in g\Lambda} \;\big|\; \xi_h \in \mathcal{H}_h \;,\; \xi_{hk^{-1}} = \rho(k)\xi_h \text{ for all } h \in g\Lambda, k \in \Lambda \big\}$$

  and with the natural Λ-$\ell^\infty(\Gamma/\Lambda)$-module structure. Note that $\mathcal{K}_{g\Lambda} \cong \mathcal{H}_g$, but again, this isomorphism depends on a choice of section $\Gamma/\Lambda \to \Gamma$.

- $\mathcal{C}_5 \to \mathcal{C}_4 : \mathcal{K} \mapsto \mathcal{H}$, with $\mathcal{H}_g = \mathcal{K}_{g\Lambda}$ and the obvious Λ-Λ-$\ell^\infty(\Gamma)$-module structure.

We say that an object $\mathcal{H} \in \mathcal{C}_5$ has finite rank if $\mathcal{H}$ is a finite dimensional Hilbert space. This is equivalent to requiring that all Hilbert spaces $\mathcal{H}_{g\Lambda}$ are finite dimensional and that there are only finitely many double cosets $\Lambda g\Lambda$ for which $\mathcal{H}_{g\Lambda}$ is nonzero. Similarly, we say that an object $\mathcal{H} \in \mathcal{C}_4$ has finite rank if all Hilbert spaces $\mathcal{H}_g$ are finite dimensional and if there are only finitely many double cosets $\Lambda g\Lambda$ for which $\mathcal{H}_g$ is nonzero. Note here that an algebraic variant of the category of finite rank objects in $\mathcal{C}_4$ was already introduced in [Z98].

In this way, we have defined the rigid C*-tensor category $\mathcal{C}_{4,f}(\Lambda < \Gamma)$ consisting of the finite rank objects in $\mathcal{C}_4$. Note that, in a different context, this rigid C*-tensor category $\mathcal{C}_{4,f}(\Lambda < \Gamma)$ already appeared in [DV10, Section 4].

Denote by $\pi : \Gamma \to G$ the canonical homomorphism. Identifying G/K and Γ/Λ and using the homomorphism $\pi : \Lambda \to K$, every K-$L^\infty(G/K)$-module $\mathcal{H}$ also is a Λ-$\ell^\infty(\Gamma/\Lambda)$-module.

This defines a functor $\mathcal{C}_2(K < G) \to \mathcal{C}_5(\Lambda < \Gamma)$ that is fully faithful because $\pi(\Lambda)$ is dense in K. Note however that this fully faithful functor need not be an equivalence of categories: an object $\mathcal{H} \in \mathcal{C}_5(\Lambda < \Gamma)$ is isomorphic with an object in the range of this functor if and only if the representation of Λ on $\mathcal{H}$ is of the form $k \mapsto \lambda(\pi(k))$ for a (necessarily unique) continuous representation λ of K on $\mathcal{H}$.

Composing with the equivalence of categories in Proposition 2.2, we have found the fully faithful C*-tensor functor $\Theta : \mathcal{C}_3(K < G) \to \mathcal{C}_4(\Lambda < \Gamma)$, sending finite rank objects to finite rank objects. By construction, Θ maps the G-$L^\infty(G/K)$-$L^\infty(G/K)$-module $L^2(G/K) \otimes L^2(G/K)$ (with G-action given by $(\lambda_g \otimes \lambda_g)_{g \in G}$ and obvious left and right $L^\infty(G/K)$-action) to the Λ-Λ-$\ell^\infty(\Gamma)$-module $\ell^2(\Gamma)$.

Next, given the outer action $\Gamma \curvearrowright^\alpha P$, we write $N = P \rtimes \Lambda$ and $M = P \rtimes \Gamma$. Consider the category $\mathrm{Bimod}(N)$ of Hilbert N-N-bimodules. We define the natural fully faithful C*-tensor functor $\mathcal{C}_4(\Lambda < \Gamma) \to \mathrm{Bimod}(N) : \mathcal{H} \mapsto \mathcal{K}$ where $\mathcal{K} = L^2(P) \otimes \mathcal{H}$ and where the N-N-bimodule structure on $\mathcal{K}$ is given by

$$(au_k) \cdot (b \otimes \xi) \cdot (du_r) = a\alpha_k(b)\alpha_{kh}(d) \otimes \lambda(k)\rho(r^{-1})\xi$$

for all $a, b, d \in P$, $k, r \in \Lambda$, $h \in \Gamma$ and $\xi \in \mathcal{H}_h$. By construction, this functor maps the Λ-Λ-$\ell^\infty(\Gamma)$-module $\ell^2(\Gamma)$ to the N-N-bimodule $L^2(M)$.

Denoting by $\mathcal{C}$ the tensor category of finite index N-N-bimodules generated by the finite index N-subbimodules of $L^2(M)$, it follows that $\mathcal{C}$ is naturally monoidally equivalent to the tensor subcategory $\mathcal{C}_0$ of $\mathcal{C}_{3,f}(K < G)$ generated by the finite rank subobjects of $L^2(G/K) \otimes L^2(G/K)$. So, it remains to prove that $\mathcal{C}_0 = \mathcal{C}_{3,f}(K < G)$. Taking the n-th tensor power of $L^2(G/K) \otimes L^2(G/K)$ and applying the equivalence between the categories $\mathcal{C}_{3,f}(K < G)$ and $\mathcal{C}_{2,f}(K < G)$, it suffices to show that every irreducible K-$L^\infty(G/K)$-module appears in one of the K-$L^\infty(G/K)$-modules $L^2(G/K) \otimes \cdots \otimes L^2(G/K)$ with diagonal G-action and action of $L^\infty(G/K)$ on the last tensor factor. Reducing with the projections 1_{gK}, this amounts to proving that for every $g \in G$, every irreducible representation of the compact group $K \cap gKg^{-1}$ appears in a tensor power of $L^2(G/K)$. Because $K < G$ is a Schlichting completion, we have that $\bigcap_{h \in G} hKh^{-1} = \{e\}$ so that the desired conclusion follows. $\square$

3 The tube algebra of $\mathcal{C}(K < G)$

Recall from [O93] the following construction of the *tube $*$-algebra* of a rigid C*-tensor category $\mathcal{C}$ (see also [GJ15, Section 3] where the terminology *annular algebra* is used, and see as well [PSV15, Section 3.3]). Whenever I is a full[5] family of objects in $\mathcal{C}$, one defines as follows the $*$-algebra $\mathcal{A}$ with underlying vector space

$$\mathcal{A} = \bigoplus_{i,j \in I, \alpha \in \mathrm{Irr}(\mathcal{C})} (i\alpha, \alpha j) .$$

Here and in what follows, we denote the tensor product in $\mathcal{C}$ by concatenation and we denote by (β, γ) the space of morphisms from γ to β. By definition, all (β, γ) are finite dimensional Banach spaces. Using the categorical traces Tr_β and Tr_γ on (β, β), resp. (γ, γ), we turn (β, γ) into a Hilbert space with scalar product

$$\langle V, W \rangle = \mathrm{Tr}_\beta(VW^*) = \mathrm{Tr}_\gamma(W^*V) .$$

[5]Fullness means that every irreducible $i \in \mathrm{Irr}(\mathcal{C})$ appears as a subobject of one of the $j \in I$.

For every $\beta \in \mathcal{C}$, the categorical trace Tr_β is defined by using a standard solution for the conjugate equations for β, i.e. morphisms $s_\beta \in (\beta\overline{\beta}, \varepsilon)$ and $t_\beta \in (\overline{\beta}\beta, \varepsilon)$ satisfying

$$(s_\beta^* \otimes 1)(1 \otimes t_\beta) = 1 \;\; , \;\; (1 \otimes s_\beta^*)(t_\beta \otimes 1) = 1 \;\; , \;\; t_\beta^*(1 \otimes V)t_\beta = s_\beta^*(V \otimes 1)s_\beta$$

for all $V \in (\beta, \beta)$. Then, $\mathrm{Tr}_\beta(V) = t_\beta^*(1 \otimes V)t_\beta = s_\beta^*(V \otimes 1)s_\beta$ and $d(\beta) = \mathrm{Tr}_\beta(1)$ is the categorical dimension of β.

We will also make use of the partial traces

$$\mathrm{Tr}_\beta \otimes \mathrm{id} : (\beta\alpha, \beta\gamma) \to (\alpha, \gamma) : (\mathrm{Tr}_\beta \otimes \mathrm{id})(V) = (t_\beta^* \otimes 1)(1 \otimes V)(t_\beta \otimes 1) \; .$$

Whenever $\mathcal{K}$ is a Hilbert space, we denote by $\mathrm{onb}(\mathcal{K})$ any choice of orthonormal basis in $\mathcal{K}$. The product in $\mathcal{A}$ is then defined as follows: for $V \in (i\alpha, \alpha j)$ and $W \in (j'\beta, \beta, k)$, the product $V \cdot W$ equals 0 when $j \neq j'$ and when $j = j'$, it is equal to

$$V \cdot W = \sum_{\gamma \in \mathrm{Irr}(\mathcal{C})} \sum_{U \in \mathrm{onb}(\alpha\beta,\gamma)} d(\gamma)\,(1 \otimes U^*)(V \otimes 1)(1 \otimes W)(U \otimes 1) \; .$$

The $*$-operation on $\mathcal{A}$ is denoted by $V \mapsto V^\#$ and defined by

$$V^\# = (t_\alpha^* \otimes 1)(1 \otimes V^* \otimes 1)(1 \otimes s_\alpha)$$

for all $V \in (i\alpha, \alpha j)$.

The $*$-algebra $\mathcal{A}$ has a natural positive faithful trace Tr and for $V \in (i\alpha, \alpha j)$, we have that $\mathrm{Tr}(V) = 0$ when $i \neq j$ or $\alpha \neq \varepsilon$, while $\mathrm{Tr}(V) = \mathrm{Tr}_i(V)$ when $i = j$ and $\alpha = \varepsilon$, so that $V \in (i, i)$.

Up to strong Morita equivalence, the tube $*$-algebra $\mathcal{A}$ does not depend on the choice of the full family I of objects in $\mathcal{C}$, see [NY15b, Theorem 3.2] and [PSV15, Section 7.2]. Also note that for an arbitrary object $\alpha \in \mathcal{C}$ and $i, j \in I$, we can associate with $V \in (i\alpha, \alpha j)$ the element in $\mathcal{A}$ given by

$$\sum_{\gamma \in \mathrm{Irr}(\mathcal{C})} \sum_{U \in \mathrm{onb}(\alpha,\gamma)} d(\gamma)\,(1 \otimes U^*)V(U \otimes 1) \; .$$

Although this map $(i\alpha, \alpha j) \to \mathcal{A}$ is not injective, we will view an element in $V \in (i\alpha, \alpha j)$ as an element of $\mathcal{A}$ in this way.

Formally allowing for infinite direct sums in $\mathcal{C}$, one defines the C*-tensor category of ind-objects in $\mathcal{C}$. Later in this section, we will only consider the rigid C*-tensor category $\mathcal{C}$ of finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules for a given totally disconnected group G with compact open subgroup $K < G$. In that case, the ind-category precisely[6] is the C*-tensor category of all G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules. Whenever $\mathcal{K}_1, \mathcal{K}_2$ are ind-objects, we denote by $(\mathcal{K}_1, \mathcal{K}_2)$ the vector space of *finitely supported* morphisms, where a morphism $V : \mathcal{K}_2 \to \mathcal{K}_1$ is said to be finitely supported if there exist projections p_i of $\mathcal{K}_i$ onto a finite dimensional subobject (i.e. an object in $\mathcal{C}$) such that $V = p_1 V = V p_2$.

We say that an ind-object $\mathcal{H}_0$ in $\mathcal{C}$ is full if every irreducible object $i \in \mathrm{Irr}(\mathcal{C})$ is isomorphic with a subobject of $\mathcal{H}_0$. We define the tube $*$-algebra of $\mathcal{C}$ with respect to a full ind-object $\mathcal{H}_0$ as the vector space

$$\mathcal{A} = \bigoplus_{\alpha \in \mathrm{Irr}(\mathcal{C})} (\mathcal{H}_0\alpha, \alpha\mathcal{H}_0)$$

[6]Using Proposition 2.2, every G-$L^\infty(G/K)$-$L^\infty(G/K)$-module is a direct sum of finite rank modules because every K-$L^\infty(G/K)$-module is a direct sum of finite dimensional modules, which follows because every unitary representation of a compact group is a direct sum of finite dimensional representations.

on which the $*$-algebra structure is defined in the same way as above. Note that $(\mathcal{H}_0, \mathcal{H}_0)$ naturally is a $*$-subalgebra of $\mathcal{A}$, given by taking $\alpha = \varepsilon$ in the above description of $\mathcal{A}$. In particular, every projection of p of $\mathcal{H}_0$ on a finite dimensional subobject of $\mathcal{H}_0$ can be viewed as a projection $p \in \mathcal{A}$. These projections serve as local units: for every finite subset $\mathcal{F} \subset \mathcal{A}$, there exists such a projection p satisfying $p \cdot V = V \cdot p$ for all $V \in \mathcal{F}$.

Whenever p_ε is the projection of $\mathcal{H}_0$ onto a copy of the trivial object ε, we identify $p_\varepsilon \cdot \mathcal{A} \cdot p_\varepsilon$ with the fusion $*$-algebra $\mathbb{C}[\mathcal{C}]$ of $\mathcal{C}$, i.e. the $*$-algebra with vector space basis $\mathrm{Irr}(\mathcal{C})$, product given by the fusion rules and $*$-operation given by the adjoint object.

To every full family I of objects in $\mathcal{C}$, we can associate the full ind-object $\mathcal{H}_0$ by taking the direct sum of all $i \in I$. The tube $*$-algebra of $\mathcal{C}$ associated with I is then naturally a $*$-subalgebra of the tube $*$-algebra of $\mathcal{C}$ associated with $\mathcal{H}_0$. If every irreducible object of $\mathcal{C}$ appears with finite multiplicity in $\mathcal{H}_0$, then this inclusion is an equality and both tube $*$-algebras are naturally isomorphic.

For the rest of this section, we fix a totally disconnected group G and a compact open subgroup $K < G$. We denote by $\mathcal{C}$ the rigid C*-tensor category of all finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules, which we denoted as $\mathcal{C}_{3,f}(K < G)$ in Section 2. We determine the tube $*$-algebra $\mathcal{A}$ of $\mathcal{C}$ with respect to the following full ind-object.

$$\begin{aligned}
&\mathcal{H}_0 = L^2(G \times G/K) \quad \text{with} \\
&(F \cdot \xi)(g, hK) = F(gK)\, \xi(g, hK) \;\; , \;\; (\xi \cdot F)(g, hK) = \xi(g, hK)\, F(ghK) \;\; \text{and} \\
&(\pi(x)\xi)(g, hK) = \xi(x^{-1}g, hK)
\end{aligned} \tag{3.1}$$

for all $\xi \in L^2(G \times G/K)$, $F \in L^\infty(G/K)$, $x, g \in G$, $hK \in G/K$. Note that every irreducible object of $\mathcal{C}$ appears with finite multiplicity in $\mathcal{H}_0$.

We denote by $(\mathrm{Ad}\, g)_{g \in G}$ the action of G on G by conjugation: $(\mathrm{Ad}\, g)(h) = ghg^{-1}$. In the rest of this paper, we will make use of the associated full and reduced C*-algebras

$$C_0(G) \rtimes^f_{\mathrm{Ad}} G \quad \text{and} \quad C_0(G) \rtimes^r_{\mathrm{Ad}} G \;\; ,$$

as well as the von Neumann algebra $L^\infty(G) \rtimes_{\mathrm{Ad}} G$. We fix the left Haar measure λ on G such that $\lambda(K) = 1$. We equip $L^\infty(G) \rtimes_{\mathrm{Ad}} G$ with the canonical normal semifinite faithful trace Tr given by

$$\mathrm{Tr}(F \lambda_f) = f(e) \int_G F(g) \Delta(g)^{-1/2}\, dg \; . \tag{3.2}$$

Note that the modular function Δ is affiliated with the center of $L^\infty(G) \rtimes_{\mathrm{Ad}} G$, so that $L^\infty(G) \rtimes_{\mathrm{Ad}} G$ need not be a factor. Also note that the measure used in (3.2) is half way between the left and the right Haar measure of G.

We consider the dense $*$-algebra $\mathrm{Pol}(L^\infty(G) \rtimes_{\mathrm{Ad}} G)$ defined as

$$\begin{aligned}
\mathrm{Pol}(L^\infty(G) \rtimes_{\mathrm{Ad}} G) = \mathrm{span}\{ 1_{\mathcal{U}}\, u_x\, p_L \mid \; & \mathcal{U} \subset G \;\text{ compact open subset} \;,\; x \in G \;, \\
& L < G \;\text{ compact open subgroup} \}
\end{aligned} \tag{3.3}$$

and where $p_L \in L(G)$ denotes the projection onto the L-invariant vectors, i.e.

$$p_L = \lambda(L)^{-1} \int_L u_k \, dk \; .$$

Note that $\mathrm{Pol}(L^\infty(G) \rtimes_{\mathrm{Ad}} G)$ equals the linear span of all $F \lambda_f$ where F and f are continuous, compactly supported, locally constant functions on G.

We now identify the tube $*$-algebra of $\mathcal{C}$ with $\mathrm{Pol}(L^\infty(G) \rtimes_{\mathrm{Ad}} G)$. For every $x \in G$ and every irreducible representation $\pi : K \cap xKx^{-1} \to \mathcal{U}(\mathcal{K})$, we denote by $\mathcal{H}(\pi, x) \in \mathrm{Irr}(\mathcal{C})$ the irreducible G-$L^\infty(G/K)$-$L^\infty(G/K)$-module such that π is isomorphic with the representation of $K \cap xKx^{-1}$ on $1_{xK} \cdot \mathcal{H}(\pi, x) \cdot 1_{eK}$. Note that this gives us the identification

$$\mathrm{Irr}(\mathcal{C}) = \{(\pi, x) \mid x \in K\backslash G/K \;,\; \pi \in \mathrm{Irr}(K \cap xKx^{-1})\} \;. \tag{3.4}$$

We denote by χ_π the character of π, i.e. the locally constant function with support $K \cap xKx^{-1}$ and $\chi_\pi(k) = \mathrm{Tr}(\pi(k))$ for all $k \in K \cap xKx^{-1}$.

Theorem 3.1. *The G-$L^\infty(G/K)$-$L^\infty(G/K)$-module $\mathcal{H}_0$ introduced in (3.1) is full. There is a natural $*$-anti-isomorphism Θ of the associated tube $*$-algebra $\mathcal{A}$ onto $\mathrm{Pol}(L^\infty(G) \rtimes_{\mathrm{Ad}} G)$. The $*$-anti-isomorphism Θ is trace preserving.*

Denoting by p_ε the projection in $\mathcal{A}$ that corresponds to the unique copy of the trivial object ε in $\mathcal{H}_0$ and identifying $p_\varepsilon \cdot \mathcal{A} \cdot p_\varepsilon$ with the fusion $*$-algebra of $\mathcal{C}$, we have that $\Theta(p_\varepsilon) = 1_K p_K$ and that the restriction of Θ to $\mathbb{C}[\mathcal{C}]$ is given by

$$d(\pi, x)^{-1} \Theta(\pi, x) = p_K \, \dim(\pi)^{-1} \chi_\pi \, u_x \, p_K \;, \tag{3.5}$$

where $d(\pi, x)$ denotes the categorical dimension of $(\pi, x) \in \mathrm{Irr}(\mathcal{C})$ and $\dim(\pi)$ denotes the ordinary dimension of the representation π.

Proof. To see that $\mathcal{H}_0$ is full, it suffices to observe that for every $h \in G$, the unitary representation of $K \cap hKh^{-1}$ on $1_{eK} \cdot \mathcal{H}_0 \cdot 1_{hK}$ contains the regular representation of $K \cap hKh^{-1}$.

Assume that $\Psi : C_0(G) \rtimes^f_{\mathrm{Ad}} G \to B(\mathcal{K})$ is any nondegenerate $*$-representation. As follows, we associate with Ψ a unitary half braiding[7] on ind-$\mathcal{C}$. Whenever $\mathcal{H}$ is a G-$L^\infty(G/K)$-$L^\infty(G/K)$-module, we consider a new G-$L^\infty(G/K)$-$L^\infty(G/K)$-module with underlying Hilbert space $\mathcal{K} \otimes \mathcal{H}$ and structure maps

$$\pi_{\mathcal{K}\otimes\mathcal{H}}(g) = \Psi(g) \otimes \pi_{\mathcal{H}}(g) \;\;,\;\; \lambda_{\mathcal{K}\otimes\mathcal{H}}(F) = (\Psi \otimes \lambda_{\mathcal{H}})\Delta(F) \;\;,\;\; \rho_{\mathcal{K}\otimes\mathcal{H}}(F) = 1 \otimes \rho_{\mathcal{H}}(F) \;\;,$$

for all $g \in G$, $F \in L^\infty(G/K)$, with $\Delta(F)(g, hK) = F(ghK)$.

We similarly turn $\mathcal{H} \otimes \mathcal{K}$ into a G-$L^\infty(G/K)$-$L^\infty(G/K)$-module with structure maps

$$\pi_{\mathcal{H}\otimes\mathcal{K}}(g) = \pi_{\mathcal{H}}(g) \otimes \Psi(g) \;\;,\;\; \lambda_{\mathcal{H}\otimes\mathcal{K}}(F) = \lambda_{\mathcal{H}}(F) \otimes 1 \;\;,\;\; \rho_{\mathcal{H}\otimes\mathcal{K}}(F) = (\rho_{\mathcal{H}} \otimes \Psi)\widetilde{\Delta}(F) \;\;,$$

where $\widetilde{\Delta}(F)(gK, h) = F(h^{-1}gK)$.

Defining the unitary $U \in M(C_0(G) \otimes K(\mathcal{H}))$ given by $U(x) = \pi_{\mathcal{H}}(x)$ for all $x \in G$ and denoting by $\Sigma : \mathcal{K} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ the flip map, one checks that $\Sigma(\Psi \otimes \mathrm{id})(U)$ is an isomorphism between the G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules $\mathcal{K} \otimes \mathcal{H}$ and $\mathcal{H} \otimes \mathcal{K}$. So, defining

$$\mathcal{K}_1 := \mathcal{K} \otimes L^2(G/K) \cong L^2(G/K) \otimes \mathcal{K} \;,$$

we have found the G-$L^\infty(G/K)$-$L^\infty(G/K)$-module $\mathcal{K}_1$ with the property that for every G-$L^\infty(G/K)$-$L^\infty(G/K)$-module $\mathcal{H}$, there is a natural unitary isomorphism

$$\sigma_{\mathcal{H}} : \mathcal{H}\mathcal{K}_1 \to \mathcal{K}_1\mathcal{H} \;.$$

[7] Formally, a unitary half braiding is an object in the Drinfeld center of ind-$\mathcal{C}$. More concretely, a unitary half braiding consists of an underlying ind-object $\mathcal{K}_1$ together with natural unitary isomorphisms $\mathcal{H}\mathcal{K}_1 \to \mathcal{K}_1\mathcal{H}$ for all objects $\mathcal{H}$. We refer to [NY15a, Section 2.1] for further details.

Here and in what follows, we denote by concatenation the tensor product in the category of G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules. So, σ is a unitary half braiding for ind-$\mathcal{C}$.

Using the ind-object $\mathcal{H}_0$ defined in (3.1) and recalling that $\mathcal{K}_1\overline{\mathcal{H}_0} = \mathcal{K} \otimes \overline{\mathcal{H}_0}$ as Hilbert spaces, we define the Hilbert space

$$\mathcal{K}_2 = (\mathcal{K} \otimes \overline{\mathcal{H}_0}, \varepsilon)$$

and we consider the tube $*$-algebra $\mathcal{A}$ associated with $\mathcal{H}_0$. Using standard solutions for the conjugate equations, there is a natural linear bijection

$$V \in (\mathcal{H}_0\mathcal{H}, \mathcal{H}\mathcal{H}_0) \mapsto \widetilde{V} \in (\mathcal{H}\overline{\mathcal{H}_0}, \overline{\mathcal{H}_0}\mathcal{H})$$

between finitely supported morphisms.

By [PSV15, Proposition 3.14] and using the partial categorical trace $\mathrm{Tr}_\mathcal{H} \otimes \mathrm{id} \otimes \mathrm{id}$, the unitary half braiding σ gives rise to a nondegenerate $*$-anti-homomorphism $\Theta : \mathcal{A} \to B(\mathcal{K}_2)$ given by

$$\Theta(V)\xi = (\mathrm{Tr}_\mathcal{H} \otimes \mathrm{id} \otimes \mathrm{id})\big((\sigma^*_\mathcal{H} \otimes 1)(1 \otimes \widetilde{V})(\xi \otimes 1)\big) \tag{3.6}$$

for all $\mathcal{H} \in \mathcal{C}$, $\xi \in \mathcal{K}_2$ and all finitely supported $V \in (\mathcal{H}_0\mathcal{H}, \mathcal{H}\mathcal{H}_0)$.

We now compute the expression in (3.6) more concretely. Whenever $h \in G$ and $K_0 < K$ is an open subgroup such that $hK_0h^{-1} \subset K$, we define the finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-module $L^2(G/K_0)_h$ with underlying Hilbert space $L^2(G/K_0)$ and structure maps

$$(x \cdot \xi)(gK_0) = \xi(x^{-1}gK_0) \quad , \quad (F_1 \cdot \xi \cdot F_2)(gK_0) = F_1(gK)\, \xi(gK_0)\, F_2(gh^{-1}K) \; .$$

Note that there is a natural isomorphism $\overline{L^2(G/K_0)_h} \cong L^2(G/K_0)_{h^{-1}}$. Letting K_0 tend to $\{e\}$, the direct limit of $L^2(G/K_0)_{h^{-1}}$ becomes $L^2(G)_{h^{-1}}$. Since $\mathcal{H}_0 = \bigoplus_{h\in G/K} L^2(G)_{h^{-1}}$, we identify

$$\overline{\mathcal{H}_0} = \bigoplus_{h\in G/K} L^2(G)_h$$

and we view $L^2(G/K_0)_h \subset \overline{\mathcal{H}_0}$ whenever $h \in G$ and $K_0 < K \cap h^{-1}Kh$ is an open subgroup.

The Hilbert space $\mathcal{K}_2$ equals the space of K-invariant vectors in $1_{eK} \cdot (\mathcal{K} \otimes \overline{\mathcal{H}_0}) \cdot 1_{eK}$. In this way, the space of K-invariant vectors in $1_{eK} \cdot (\mathcal{K} \otimes L^2(G/K_0)_h) \cdot 1_{eK}$ naturally is a subspace of $\mathcal{K}_2$. But this last space of K-invariant vectors can be unitarily identified with $\Psi(1_{Kh^{-1}}\, p_{hK_0h^{-1}})\mathcal{K}$ by sending the vector $\xi_0 \in \Psi(1_{Kh^{-1}}\, p_{hK_0h^{-1}})\mathcal{K}$ to the vector

$$\Delta(h)^{-1/2} \sum_{k\in K/hK_0h^{-1}} \Psi(k)\xi_0 \otimes 1_{khK_0} \in \mathcal{K} \otimes L^2(G/K_0) \; .$$

We now use that for every $\mathcal{H} \in \mathcal{C}$, the categorical trace $\mathrm{Tr}_\mathcal{H}$ on $(\mathcal{H}, \mathcal{H})$ is given by

$$\begin{aligned}\mathrm{Tr}_\mathcal{H}(V) &= \sum_{x\in G/K, \eta\in \mathrm{onb}(1_{xK}\cdot\mathcal{H}\cdot 1_{eK})} \Delta(x)^{-1/2}\, \langle V\eta, \eta\rangle \\ &= \sum_{y\in G/K, \eta\in \mathrm{onb}(1_{eK}\cdot\mathcal{H}\cdot 1_{yK})} \Delta(y)^{-1/2}\, \langle V\eta, \eta\rangle \; .\end{aligned}$$

A straightforward computation then gives that for all $\mathcal{H} \in \mathcal{C}$ and all

$$V \in \big(\, \overline{L^2(G/K_0)_g}\, \mathcal{H} \; , \; \mathcal{H}\, \overline{L^2(G/K_1)_h} \,\big)$$

with $g,h \in G$ and $K_0 < K \cap g^{-1}Kg$, $K_1 < K \cap h^{-1}Kh$ open subgroups, we have

$$\Theta(V) = \Delta(g)^{-1/2}\,\Delta(h)^{1/2}\,[K:K_1] \sum_{\substack{x\in G/gK_0g^{-1}\\ y\in K/K_2\\ \eta\in \mathrm{onb}(1_{xK}\cdot\mathcal{H}\cdot 1_{eK})}} \Delta(x)^{-1/2}\,\Psi(1_{K_2y^{-1}h^{-1}}\,u_x\,p_{gK_0g^{-1}})\,\langle \widetilde{V}(1_{xgK_0}\otimes\eta), \pi_{\mathcal{H}}(hy)\eta\otimes 1_{hK_1}\rangle\;, \tag{3.7}$$

whenever $K_2 < K$ is a small enough open subgroup such that $\pi_{\mathcal{H}}(k)$ is the identity on $\mathcal{H}\cdot 1_{eK}$ for all $k\in K_2$. Note that because $\mathcal{H}$ has finite rank, such an open subgroup K_2 exists. Also, there are only finitely many $x \in G/K$ such that $1_{xK}\cdot\mathcal{H}\cdot 1_{eK}$ is nonzero. Therefore, the sum appearing in (3.7) is finite.

Applying this to the regular representation $C_0(G)\rtimes^f_{\mathrm{Ad}} G \to B(L^2(G\times G))$, we see that (3.7) provides a $*$-anti-homomorphisms Θ from $\mathcal{A}$ to $\mathrm{Pol}(L^\infty(G)\rtimes_{\mathrm{Ad}} G)$. A direct computation gives that Θ is trace preserving, using the trace Tr on $L^\infty(G)\rtimes_{\mathrm{Ad}} G$ defined in (3.2). In particular, Θ is injective.

We now prove that Θ is surjective. Fix elements $g,h,\alpha\in G$ satisfying $\alpha g = h\alpha$. Choose any open subgroup $K_0 < K$ such that gK_0g^{-1}, $\alpha K_0\alpha^{-1}$ and $K_1 := h^{-1}\alpha K_0\alpha^{-1}h$ are all subgroups of K. Put $\mathcal{H} = L^2(G/K_0)_\alpha$ and note that $\mathcal{H}$, $L^2(G/K_0)_g$ and $L^2(G/K_1)_h$ are well defined objects in $\mathcal{C}$. For every $k\in K$, we consider the vectors

$$\begin{aligned} 1_{k\alpha g K_0}\otimes 1_{k\alpha K_0} &\in 1_{k\alpha g K}\cdot\big(L^2(G/K_0)_g\,\mathcal{H}\big)\cdot 1_{eK} \quad\text{and}\\ 1_{kh\alpha K_0}\otimes 1_{khK_1} &\in 1_{k\alpha g K}\cdot\big(\mathcal{H}\,L^2(G/K_1)_h\big)\cdot 1_{eK}\;. \end{aligned}$$

In both cases, we get an orthogonal family of vectors indexed by

$$k \in K/(K\cap \alpha K_0\alpha^{-1}\cap \alpha g K_0(\alpha g)^{-1})\;.$$

So, we can uniquely define $V \in \big(\,\overline{L^2(G/K_0)_g}\,\mathcal{H}\;,\;\mathcal{H}\,\overline{L^2(G/K_1)_h}\,\big)$ such that the restriction of $\widetilde{V}$ to $\big(L^2(G/K_0)_g\,\mathcal{H}\big)\cdot 1_{eK}$ is the partial isometry given by

$$1_{k\alpha g K_0}\otimes 1_{k\alpha K_0} \mapsto \Delta(\alpha)^{-1/2}\,\Delta(h)^{-1/2}\,1_{kh\alpha K_0}\otimes 1_{khK_1}\quad\text{for all}\;\; k\in K\;.$$

A direct computation gives that $\Theta(V)$ is equal to a nonzero multiple of

$$1_{\alpha K_0\alpha^{-1}h^{-1}}\,u_\alpha\,p_{gK_0g^{-1}}\;. \tag{3.8}$$

From (3.7), we also get that Θ maps $(\mathcal{H}_0,\mathcal{H}_0)\subset\mathcal{A}$ onto $\mathrm{Pol}(L^\infty(K\backslash G)\rtimes K)$, defined as the linear span of all

$$1_{Kx}\,u_k\,p_L$$

with $x\in G$, $k\in K$ and $L<K$ an open subgroup. In combination with (3.8), it follows that Θ is surjective.

Finally, by restricting (3.7) to the cases where $g=h=e$ and $K_0=K_1=K$, we find that (3.5) holds. ∎

We recall from [PV14] the notion of a *completely positive (cp) multiplier* on a rigid C*-tensor category $\mathcal{C}$. By [PV14, Proposition 3.6], to every function $\varphi:\mathrm{Irr}(\mathcal{C})\to\mathbb{C}$ is associated a system of linear maps

$$\Psi^\varphi_{\alpha_1|\beta_1,\alpha_2|\beta_2} : (\alpha_1\beta_1,\alpha_2\beta_2)\to(\alpha_1\beta_1,\alpha_2\beta_2)\quad\text{for all}\;\;\alpha_i,\beta_i\in\mathcal{C} \tag{3.9}$$

satisfying

$$\Psi^{\varphi}_{\alpha_3|\beta_3,\alpha_4|\beta_4}((X\otimes Y)V(Z\otimes T)) = (X\otimes Y)\,\Psi^{\varphi}_{\alpha_1|\beta_1,\alpha_2|\beta_2}(V)\,(Z\otimes T)$$

for all $X\in(\alpha_3,\alpha_1)$, $Y\in(\beta_3,\beta_1)$, $Z\in(\alpha_2,\alpha_4)$, $T\in(\beta_2,\beta_4)$, as well as

$$\begin{aligned}&\Psi^{\varphi}_{\alpha|\overline{\alpha},\varepsilon|\varepsilon}(s_\alpha) = \varphi(\alpha)\,s_\alpha \quad \text{and}\\ &\Psi^{\varphi}_{\alpha_1\alpha_2|\beta_2\beta_1,\alpha_3\alpha_4|\beta_4\beta_3}(1\otimes V\otimes 1) = 1\otimes\Psi^{\varphi}_{\alpha_2|\beta_2,\alpha_4|\beta_4}(V)\otimes 1\end{aligned}$$

for all $V\in(\alpha_2\beta_2,\alpha_4\beta_4)$.

Definition 3.2 ([PV14, Definition 3.4])**.** Let $\mathcal{C}$ be a rigid C*-tensor category.

- A *cp-multiplier* on $\mathcal{C}$ is a function $\varphi:\mathrm{Irr}(\mathcal{C})\to\mathbb{C}$ such that the maps $\Psi^{\varphi}_{\alpha|\beta,\alpha|\beta}$ on $(\alpha\beta,\alpha\beta)$ are completely positive for all $\alpha,\beta\in\mathcal{C}$.
- A cp-multiplier $\varphi:\mathrm{Irr}(\mathcal{C})\to\mathbb{C}$ is said to be c_0 if the function $\varphi:\mathrm{Irr}(\mathcal{C})\to\mathbb{C}$ tends to zero at infinity.
- A *cb-multiplier* on $\mathcal{C}$ is a function $\varphi:\mathrm{Irr}(\mathcal{C})\to\mathbb{C}$ such that

$$\|\varphi\|_{\mathrm{cb}} := \sup_{\alpha_i,\beta_i\in\mathcal{C}} \|\Psi^{\varphi}_{\alpha_1|\beta_1,\alpha_2|\beta_2}\|_{\mathrm{cb}} < \infty\,.$$

A function $\varphi:\mathrm{Irr}(\mathcal{C})\to\mathbb{C}$ gives rise to the following linear functional $\omega_\varphi:\mathcal{A}\to\mathbb{C}$ on the tube algebra $\mathcal{A}$ of $\mathcal{C}$ with respect to any full family of objects containing once the trivial object ε:

$$\omega_\varphi:\mathcal{A}\to\mathbb{C}:\omega_\varphi(V)=\begin{cases} d(\alpha)\,\varphi(\alpha) & \text{if } V=1_\alpha\in(\varepsilon\alpha,\alpha\varepsilon),\\ 0 & \text{if } V\in(i\alpha,\alpha j) \text{ with } i\neq\varepsilon \text{ or } j\neq\varepsilon.\end{cases}$$

By [GJ15, Theorem 6.6], the function $\varphi:\mathrm{Irr}(\mathcal{C})\to\mathbb{C}$ is a cp-multiplier in the sense of Definition 3.2 if and only if ω_φ is positive on $\mathcal{A}$ in the sense that $\omega_\varphi(V\cdot V^\#)\geq 0$ for all $V\in\mathcal{A}$. In Proposition 5.1, we prove a characterization of cb-multipliers in terms of completely bounded multipliers of the tube $*$-algebra.

From Theorem 3.1, we then get the following result. We again denote by $\mathcal{C}$ be the rigid C*-tensor category of finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules and we identify $\mathrm{Irr}(\mathcal{C})$ as in (3.4) with the set of pairs (π,x) where $x\in K\backslash G/K$ and π is an irreducible representation of the compact group $K\cap xKx^{-1}$. In order to identify the c_0 cp-multipliers on $\mathcal{C}$, we introduce the following definition.

Definition 3.3. We say that a complex measure μ on G (i.e. an element of $C_0(G)^*$) is c_0 if

$$\lambda(\mu) := \int_G \lambda_g\, d\mu(g)\in L(G)$$

belongs to $C^*_r(G)$.

We say that a positive functional ω on $C_0(G)\rtimes^f_{\mathrm{Ad}} G$ is c_0 if for every $x\in G$, the complex measure μ_x defined by $\mu_x(F)=\omega(Fu_x)$ for all $F\in C_0(G)$ is c_0 and if the function $G\to C^*_r(G): x\mapsto\lambda(\mu_x)$ tends to zero at infinity, i.e. $\lim_{x\to\infty}\|\lambda(\mu_x)\|=0$.

Proposition 3.4. *The formula*

$$\varphi(\pi, x) = \omega(p_K \dim(\pi)^{-1} \chi_\pi \, u_x \, p_K) \tag{3.10}$$

gives a bijection between the cp-multipliers φ on $\mathrm{Irr}(\mathcal{C})$ and the positive functionals ω on the C^*-algebra $q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$, where $q = 1_K p_K$.

The cp-multiplier φ is c_0 if and only if the positive functional ω is c_0 in the sense of Definition 3.3.

Using the notations $C_u(\mathcal{C})$ and $C_r(\mathcal{C})$ of [PV14, Definition 4.1] for the universal and reduced C^*-algebra of $\mathcal{C}$, we have the natural anti-isomorphisms $C_u(\mathcal{C}) \to q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$ and $C_r(\mathcal{C}) \to q(C_0(G) \rtimes^r_{\mathrm{Ad}} G)q$.

Proof. Note that the G-$L^\infty(G/K)$-$L^\infty(G/K)$-module $\mathcal{H}_0$ in (3.1) contains exactly once the trivial module. The first part of the proposition is then a direct consequence of Theorem 3.1 and the above mentioned characterization [GJ15] of cp-multipliers as positive functionals on the tube $*$-algebra. The isomorphisms for $C_u(\mathcal{C})$ and $C_r(\mathcal{C})$ follow in the same way.

Fix a positive functional ω on $q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$ with corresponding cp-multiplier $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ given by (3.10). We extend ω to $C_0(G) \rtimes^f_{\mathrm{Ad}} G$ by $\omega(T) = \omega(qTq)$. For every $x \in G$, define $\mu_x \in C_0(G)^*$ given by $\mu_x(F) = \omega(F u_x)$ for all $F \in C_0(G)$. Note that μ_x is supported on $K \cap xKx^{-1}$ and that μ_x is $\mathrm{Ad}(K \cap xKx^{-1})$-invariant. Therefore, $\lambda(\mu_x) \in \mathcal{Z}(L(K \cap xKx^{-1}))$. For every $\pi \in \mathrm{Irr}(K \cap xKx^{-1})$, denote by $z_\pi \in \mathcal{Z}(L(K \cap xKx^{-1}))$ the corresponding minimal central projection. From (3.10), we get that

$$\lambda(\mu_x) z_\pi = \varphi(\pi, x) z_\pi \quad \text{for all } \; x \in G \; , \; \pi \in \mathrm{Irr}(K \cap xKx^{-1}) \; . \tag{3.11}$$

For a fixed $x \in G$, an element $T \in \mathcal{Z}(L(K \cap xKx^{-1}))$ belongs to $C_r^*(G)$ if and only if $T \in C_r^*(K \cap xKx^{-1})$ if and only if $\lim_{\pi \to \infty} \|T z_\pi\| = 0$. Also, $\|T\| = \sup_{\pi \in \mathrm{Irr}(K \cap xKx^{-1})} \|T z_\pi\|$. So by (3.11), we get that μ_x is c_0 if and only if

$$\lim_{\pi \to \infty} |\varphi(\pi, x)| = 0 \tag{3.12}$$

and that ω is a c_0 functional if and only if (3.12) holds for all $x \in G$ and we moreover have that

$$\lim_{x \to \infty} \bigl(\sup_{\pi \in \mathrm{Irr}(K \cap xKx^{-1})} |\varphi(\pi, x)| \bigr) = 0 \; .$$

Altogether, it follows that ω is a c_0 functional in the sense of Definition 3.3 if and only if φ is a c_0-function. □

For later use, we record the following lemma.

Lemma 3.5. *Let μ be a probability measure on G that is c_0 in the sense of Definition 3.3. Then every complex measure $\omega \in C_0(G)^*$ that is absolutely continuous with respect to μ is still c_0.*

Proof. Denote by $C_c(G)$ the space of continuous compactly supported functions on G. Since $C_c(G) \subset L^1(G, \mu)$ is dense, it is sufficient to prove that $F \cdot \mu$ is c_0 for every $F \in C_c(G)$. Denote by $\omega_F \in C_r^*(G)^*$ the functional determined by $\omega_F(\lambda_x) = F(x)$ for all $x \in G$. Denote by $\widehat{\Delta} : C_r^*(G) \to M(C_r^*(G) \otimes C_r^*(G))$ the comultiplication determined by $\widehat{\Delta}(\lambda_x) = \lambda_x \otimes \lambda_x$. Recall that for every $\omega \in C_r^*(G)^*$ and every $T \in C_r^*(G)$, we have that $(\omega \otimes \mathrm{id})\widehat{\Delta}(T) \in C_r^*(G)$. Since

$$\lambda(F \cdot \mu) = (\omega_F \otimes \mathrm{id})\widehat{\Delta}(\lambda(\mu)) \; ,$$

the lemma is proven. □

4 Haagerup property and property (T) for $\mathcal{C}(K < G)$

In Definition 3.2, we already recalled the notion of a cp-multiplier $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ on a rigid C*-tensor category $\mathcal{C}$. In terms of cp-multipliers, *amenability* of a rigid C*-tensor category, as defined in [P94a, LR96], amounts to the existence of finitely supported cp-multipliers $\varphi_n : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ that converge to 1 pointwise, see [PV14, Proposition 5.3]. Following [PV14, Definition 5.1], a rigid C*-tensor category $\mathcal{C}$ has the *Haagerup property* if there exist c_0 cp-multipliers $\varphi_n : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ that converge to 1 pointwise, while $\mathcal{C}$ has *property (T)* if all cp-multipliers converging to 1 pointwise, must converge to 1 uniformly.

Similarly, when $\mathcal{C}_1$ is a full C*-tensor subcategory of $\mathcal{C}$, we say that $\mathcal{C}_1 \subset \mathcal{C}$ has the *relative property (T)* if all cp-multipliers on $\mathcal{C}$ converging to 1 pointwise, must converge to 1 uniformly on $\mathrm{Irr}(\mathcal{C}_1) \subset \mathrm{Irr}(\mathcal{C})$.

We now turn back to the rigid C*-tensor category $\mathcal{C}$ of finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules, where G is a totally disconnected group G and $K < G$ is a compact open subgroup. Note that $\mathrm{Rep}\, K$ is a full C*-tensor subcategory of $\mathcal{C}$, consisting of the G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules $\mathcal{H}$ with the property that $1_{xK} \cdot \mathcal{H} \cdot 1_{eK}$ is zero for all $x \notin K$.

Recall from Definition 3.3 the notion of a c_0 complex measure on G. We identify the space of complex measures with $C_0(G)^*$ and we denote by $\mathcal{S}(C_0(G)) \subset C_0(G)^*$ the state space of $C_0(G)$, i.e. the set of probability measures on G.

Theorem 4.1. *Let G be a totally disconnected group and $K < G$ a compact open subgroup. Denote by $\mathcal{C}$ the rigid C*-tensor category of finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules.*

1. *$\mathcal{C}$ is amenable if and only if G is amenable.*

2. *$\mathcal{C}$ has the Haagerup property if and only if G has the Haagerup property and there exists a sequence of c_0 probability measures $\mu_n \in \mathcal{S}(C_0(G))$ such that $\mu_n \to \delta_e$ weakly* and such that $\|\mu_n \circ \mathrm{Ad}\, x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$.*

3. *$\mathcal{C}$ has property (T) if and only if G has property (T) and every sequence sequence of $\mathrm{Ad}\, G$-invariant probability measures $\mu_n \in \mathcal{S}(C_0(G))$ that converges to δ_e weakly* must converge in norm.*

4. *$\mathrm{Rep}\, K \subset \mathcal{C}$ has the relative property (T) if and only if every sequence of probability measures $\mu_n \in \mathcal{S}(C_0(G))$ such that $\mu_n \to \delta_e$ weakly* and $\|\mu_n \circ \mathrm{Ad}\, x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$ satisfies $\|\mu_n - \delta_e\| \to 0$.*

Proof. Denote by $\epsilon : C_0(G) \rtimes^f_{\mathrm{Ad}} G \to \mathbb{C}$ the character given by $\epsilon(F\lambda_f) = F(e) \int_G f(x)\, dx$. Write $q = 1_K p_K$.

1. Combining Proposition 3.4 and [PV14, Proposition 5.3], we get that $\mathcal{C}$ is amenable if and only if the canonical $*$-homomorphism $q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q \to q(C_0(G) \rtimes^r_{\mathrm{Ad}} G)q$ is an isomorphism. This holds if and only if G is amenable.

2. First assume that $\mathcal{C}$ has the Haagerup property. By Proposition 3.4, we find a sequence of states ω_n on $q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$ such that $\omega_n \to \epsilon$ weakly* and such that every ω_n is a c_0 state in the sense of Definition 3.3. For every $x \in G$, define $\mu_n(x) \in C_0(G)^*$ given by $\mu_n(x)(F) = \omega_n(F u_x)$.

Using the strictly continuous extension of ω_n to the multiplier algebra $M(C_0(G) \rtimes^f_{\mathrm{Ad}} G)$, we get that $x \mapsto \omega_n(u_x)$ is a sequence of continuous positive definite functions converging to 1 uniformly on compact subsets of G. We claim that for every fixed n, the function $x \mapsto \omega_n(x)$

tends to 0 at infinity. Denote by $\epsilon_K : C^*_r(G) \to \mathbb{C}$ the state given by composing the conditional expectation $C^*_r(G) \to C^*_r(K)$ with the trivial representation $\epsilon : C^*_r(K) \to \mathbb{C}$. Then,

$$\omega_n(x) = \epsilon_K(\lambda(\mu_n(x)))$$

and the claim is proven. So, G has the Haagerup property.

The restriction of ω_n to $C_0(G)$ provides a sequence of c_0 probability measures $\mu_n \in \mathcal{S}(C_0(G))$ such that $\mu_n \to \delta_e$ weakly* and $\|\mu_n \circ \operatorname{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$.

Conversely assume that G has the Haagerup property and that μ_n is such a sequence of probability measures. By restricting μ_n to K, normalizing and integrating $\int_K (\mu_n \circ \operatorname{Ad} k)\, dk$, we may assume that the probability measures μ_n are supported on K and are $\operatorname{Ad} K$-invariant. Fix a strictly positive right K-invariant function $w : G \to \mathbb{R}^+_0$ with $\int_G w(g)\, dg = 1$. Define the probability measures $\widetilde{\mu}_n$ on G given by

$$\widetilde{\mu}_n = \int_G w(g)\, \mu_n \circ \operatorname{Ad} g \; dg \; .$$

Note that $\widetilde{\mu}_n$ is still $\operatorname{Ad} K$-invariant. Also,

$$\lambda(\widetilde{\mu}_n) = \int_G w(g)\, \lambda_g^*\, \lambda(\mu_n)\, \lambda_g \; dg$$

so that each $\widetilde{\mu}_n$ is a c_0 probability measure.

By construction, for every $x \in G$, the measure $\widetilde{\mu}_n \circ \operatorname{Ad} x$ is absolutely continuous with respect to $\widetilde{\mu}_n$. We denote by $\Delta_n(x)$ the Radon-Nikodym derivative and define the unitary representations

$$\theta_n : G \to \mathcal{U}(L^2(G, \widetilde{\mu}_n)) : \theta_n(x)\xi = \Delta_n(x)^{1/2}\, \xi \circ \operatorname{Ad} x^{-1} \; .$$

We also define $\theta_n : C_0(G) \to B(L^2(G, \widetilde{\mu}_n))$ given by multiplication operators and we have thus defined a nondegenerate $*$-representation of $C_0(G) \rtimes^f_{\operatorname{Ad}} G$ on $L^2(G, \widetilde{\mu}_n)$.

Note that μ_n is absolutely continuous with respect to $\widetilde{\mu}_n$. We denote by $\zeta_n \in L^2(G, \widetilde{\mu}_n)$ the square root of the Radon-Nikodym derivative of μ_n with respect to $\widetilde{\mu}_n$. Since both μ_n and $\widetilde{\mu}_n$ are $\operatorname{Ad} K$-invariant, we get that $\theta_n(p_K)\zeta_n = \zeta_n$. Since μ_n is supported on K, also ζ_n is supported on K meaning that $\theta(1_K)\zeta_n = \zeta_n$.

Since G has the Haagerup property, we can also fix a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ and a sequence of $\pi(K)$-invariant unit vectors $\xi_n \in \mathcal{H}$ such that $\|\pi(x)\xi_n - \xi_n\| \to 0$ uniformly on compact sets of $x \in G$ and, for every fixed n, the function $x \mapsto \langle \pi(x)\xi_n, \xi_n \rangle$ tends to zero at infinity.

The formulas $\psi(x) = \theta_n(x) \otimes \pi(x)$ and $\psi(F) = \theta(F) \otimes 1$ define a nondegenerate $*$-representation of $C_0(G) \rtimes^f_{\operatorname{Ad}} G$ on $L^2(G, \widetilde{\mu}_n) \otimes \mathcal{H}$. We define the states ω_n on $C_0(G) \rtimes^f_{\operatorname{Ad}} G$ given by $\omega_n(T) = \langle \psi(T)(\zeta_n \otimes \xi_n), \zeta_n \otimes \xi_n \rangle$. By construction, $\omega_n(q) = 1$ for all n and $\omega_n \to \epsilon$ weakly*. It remains to prove that each ω_n is a c_0 state. Proposition 3.4 then gives that $\mathcal{C}$ has the Haagerup property.

Fix n. Defining $\mu_n(x) \in C_0(G)^*$ given by $\mu_n(x)(F) = \omega_n(F u_x)$, we get that

$$\mu_n(x)(F) = \langle \theta_n(F)\, \theta(x)\, \zeta_n, \zeta_n \rangle \; \langle \pi(x)\xi_n, \xi_n \rangle \; .$$

Since the function $x \mapsto \langle \pi(x)\xi_n, \xi_n \rangle$ tends to zero at infinity, we get that even $x \mapsto \|\mu_n(x)\|$ tends to zero at infinity. So, we only have to show that for every fixed x, the complex measure given by $F \mapsto \langle \theta_n(F)\, \theta(x)\, \zeta_n, \zeta_n \rangle$ is c_0. By construction, this complex measure is absolutely continuous with respect to $\widetilde{\mu}_n$. The conclusion then follows from Lemma 3.5.

3. Note that it follows from [PV14, Proposition 5.5] that $\mathcal{C}$ has property (T) if and only if every sequence of states on $q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$ converging weakly* to ϵ must converge to ϵ in norm.

First assume that $\mathcal{C}$ has property (T). Both states on $C^*(G)$ and $\operatorname{Ad} G$-invariant states on $C_0(G)$ give rise to states on $C_0(G) \rtimes^f_{\mathrm{Ad}} G$. One implication of 3 thus follows immediately. Conversely assume that G has property (T) and that every sequence of $\operatorname{Ad} G$-invariant probability measures $\mu_n \in \mathcal{S}(C_0(G))$ converging weakly* to δ_e must converge in norm to δ_e. Let ω_n be a sequence of states on $q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$ converging to ϵ weakly*. Let $p \in C^*(G)$ be the Kazhdan projection. Replacing ω_n by $\omega_n(p)^{-1}\, p \cdot \omega_n \cdot p$, we may assume that ω_n is left and right G-invariant. This means that $\omega_n(F u_x) = \mu_n(F)$ for all $F \in C_0(G)$, $x \in G$, where μ_n is a sequence of $\operatorname{Ad} G$-invariant probability measures on G converging weakly* to δ_e. Thus $\|\mu_n - \delta_e\| \to 0$ so that $\|\omega_n - \epsilon\| \to 0$.

4. First assume that $\operatorname{Rep} K \subset \mathcal{C}$ has the relative property (T) and take a sequence of probability measures $\mu_n \in \mathcal{S}(C_0(G))$ such that $\mu_n \to \delta_e$ weakly* and $\|\mu_n \circ \operatorname{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$. We must prove that $\|\mu_n - \delta_e\| \to 0$. As in the proof of 2, we may assume that μ_n is supported on K and that μ_n is $\operatorname{Ad} K$-invariant, so that we can construct a sequence of states ω_n on $C_0(G) \rtimes^f_{\mathrm{Ad}} G$ such that $\omega_n \to \delta_e$ weakly*, $\omega_n = q \cdot \omega_n \cdot q$ and $\omega_n|_{C_0(G)} = \mu_n$ for all n.

The formula (3.10) associates to ω_n a sequence of cp-multipliers φ_n on $\mathcal{C}$ converging to 1 pointwise. Since $\operatorname{Rep} K \subset \mathcal{C}$ has the relative property (T), we conclude that $\varphi_n(\pi, e) \to 1$ uniformly on $\pi \in \operatorname{Irr}(K)$. Using [PV14, Lemma 5.6], it follows that $\|\omega_n|_{C_0(G)} - \delta_e\| \to 0$. So, $\|\mu_n - \delta_e\| \to 0$.

To prove the converse, let $\varphi_n : \operatorname{Irr}(\mathcal{C}) \to \mathbb{C}$ be a sequence of cp-multipliers on $\mathcal{C}$ converging to 1 pointwise. Denote by ω_n the states on $q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$ associated with φ_n in Proposition 3.4. Since $\omega_n \to \epsilon$ weakly*, the restriction $\mu_n := \omega_n|_{C_0(G)}$ is a sequence of probability measures on G such that $\mu_n \to \delta_e$ weakly* and $\|\mu_n \circ \operatorname{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$. By our assumption, $\|\mu_n - \delta_e\| \to 0$. For every $\pi \in \operatorname{Irr}(K)$, the function $\dim(\pi)^{-1}\chi_\pi$ has norm 1. Therefore, $\omega_n(\dim(\pi)^{-1}\chi_\pi) \to 1$ uniformly on $\operatorname{Irr}(K)$. By (3.10), this means that $\varphi_n \to 1$ uniformly on $\operatorname{Irr}(K)$. □

The following proposition gives a concrete example where G has the Haagerup property, while $\mathcal{C}(K < G)$ does not and even has $\operatorname{Rep} K$ as a full C*-tensor subcategory with the relative property (T).

Proposition 4.2. *Let F be a non-archimedean local field with characteristic $\neq 2$. Let $k \geq 2$ and define $G = \mathrm{SL}(k, F)$. Let $K < G$ be a compact open subgroup, e.g. $K = \mathrm{SL}(k, \mathcal{O})$, where $\mathcal{O}$ is the ring of integers of F. Denote by $\mathcal{C}$ the rigid C*-tensor category of finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules.*

1. *$\operatorname{Rep} K \subset \mathcal{C}$ has the relative property (T). In particular, $\mathcal{C}$ does not have the Haagerup property, although for $k = 2$, the group G has the Haagerup property.*
2. *$\mathcal{C}$ has property (T) for all $k \geq 3$.*

Proof. We denote by $\mathbb{I}$ the identity element of $G = \mathrm{SL}(k, F)$. Let $\mu_n \in \mathcal{S}(C_0(G))$ be a sequence of probability measures on G such that $\mu_n \to \delta_{\mathbb{I}}$ weakly* and $\|\mu_n \circ \operatorname{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$. Assume that $\|\mu_n - \delta_{\mathbb{I}}\| \not\to 0$. Passing to a subsequence and replacing μ_n by the normalization of $\mu_n - \mu_n(\{\mathbb{I}\})\delta_{\mathbb{I}}$, we may assume that $\mu_n(\{\mathbb{I}\}) = 0$ for all n. Since $\mu_n \to \delta_{\mathbb{I}}$ weakly* and since there are at most k of k'th roots of unity in F, we may also assume that $\mu_n(\{\lambda\mathbb{I}\}) = 0$ for all n and all k'th roots of unity $\lambda \in F$.

Every μ_n defines a state Ω_n on the C*-algebra $\mathcal{L}(G)$ of all bounded Borel functions on G. Choose a weak*-limit point $\Omega \in \mathcal{L}(G)^*$ of the sequence (Ω_n). Then, Ω induces an $\operatorname{Ad} G$-invariant mean on the Borel sets of G. In particular, Ω defines an $\operatorname{Ad} G$-invariant mean Ω on the Borel sets of the space $M_n(F)$ of $n \times n$ matrices over F. By Lemma 4.5 below, Ω is supported on the diagonal $F\mathbb{I} \subset M_n(F)$. Since Ω is also supported on G, it follows that Ω is supported on the finite set of $\lambda\mathbb{I}$ where λ is a k'th root of unity in F. But by construction, $\Omega(\{\lambda\mathbb{I}\}) = 0$ for all k'th roots of unity $\lambda \in F$. We have reached a contradiction. So, $\|\mu_n - \delta_{\mathbb{I}}\| \to 0$.

By Theorem 4.1, $\operatorname{Rep} K \subset \mathcal{C}$ has the relative property (T). For $k \geq 3$, the group $\mathrm{SL}(k, F)$ has property (T) and it follows from Theorem 4.1 that $\mathcal{C}$ has property (T). □

The following example of [C05] illustrates that G may have property (T), while the category $\mathcal{C}$ of finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules does not.

Example 4.3. Let F be a non-archimedean local field and $k \geq 3$. Define the closed subgroup $G < \mathrm{SL}(k+2, F)$ given by

$$G = \left\{ \begin{pmatrix} 1 & b_1 & \cdots & b_k & c \\ 0 & a_{11} & \cdots & a_{1k} & d_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{k1} & \cdots & a_{kk} & d_k \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \middle| A = (a_{ij}) \in \mathrm{SL}(k, F) \;,\; b_i, c, d_j \in F \right\} .$$

As in [C05], we get that G has property (T). Also, the center of G is isomorphic with F (sitting in the upper right corner) and since F is non discrete, we can take a sequence $g_n \in \mathcal{Z}(G)$ with $g_n \neq e$ for all n and $g_n \to e$. Using the $\operatorname{Ad} G$-invariant probability measures δ_{g_n}, it follows from Theorem 4.1 that $\mathcal{C}$ does not have property (T).

Finally, we also include a nonamenable example having the Haagerup property.

Example 4.4. Let $2 \leq |m| < n$ be integers. Define the totally disconnected compact abelian group $K = \mathbb{Z}_{nm}$ as the profinite completion of $\mathbb{Z}$ with respect to the decreasing sequence of finite index subgroups $(n^k m^k \mathbb{Z})_{k \geq 0}$. We have open subgroups $mK < K$ and $nK < K$, as well as the isomorphism $\varphi : mK \to nK : \varphi(mk) = nk$ for all $k \in K$. We define G as the HNN extension of K and φ. Alternatively, we may view $K < G$ as the Schlichting completion of the Baumslag-Solitar group

$$B(m, n) = \langle a, t \mid t a^m t^{-1} = a^n \rangle$$

and the almost normal subgroup $\langle a \rangle$.

Since G is acting properly on a tree, G has the Haagerup property. Also, G is nonamenable. For all positive integers $k, l \geq 0$, we denote by $\mu_{k,l}$ the normalized Haar measure on the open subgroup $n^k m^l K$. Note that $\varphi_*(\mu_{k,l}) = \mu_{k+1,l-1}$ whenever $k, l \geq 1$. Then the probability measures

$$\mu_n := \frac{1}{n+1} \sum_{k=0}^{n} \mu_{n+k, 2n-k}$$

are absolutely continuous with respect to the Haar measure of G, and thus c_0 in the sense of Definition 3.3, and they satisfy $\mu_n \to \delta_e$ weakly* and $\|\mu_n \circ \operatorname{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$. By Theorem 4.1, $\mathcal{C}$ has the Haagerup property.

Lemma 4.5. *Let F be a local field with characteristic $\neq 2$. Let $k \geq 2$ and define $G = \mathrm{SL}(k, F)$. Every $\operatorname{Ad} G$-invariant mean on the Borel sets of the space $M_k(F)$ of $k \times k$ matrices over F is supported on the diagonal $F\mathbb{I} \subset M_k(F)$.*

Proof. We start by proving the lemma for $k = 2$. So assume that m is an $\operatorname{Ad}\mathrm{SL}(2, F)$-invariant mean on the Borel sets of $M_2(F)$.

In the proof of [BHV08, Proposition 1.4.12], it is shown that if m is a mean on the Borel sets of F^2 that is invariant under the transformations $\lambda \cdot (x, y) := (x + \lambda y, y)$ for all $\lambda \in F$, then

$$m(\{(x, y) \mid (x, y) \neq (0, 0), |x| \leq |y|\}) = 0 .$$

Define $g_\lambda := \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ and notice that

$$g_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} g_\lambda^{-1} = \begin{pmatrix} a + \lambda c & -\lambda a + b - \lambda^2 c + \lambda d \\ c & -\lambda c + d \end{pmatrix} .$$

Hence, the map $\theta : M_2(F) \to F^2 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a - d, c)$ satisfies $\theta(g_\lambda A g_\lambda^{-1}) = (2\lambda) \cdot \theta(A)$. Therefore, $m(\Omega_0) = 0$ for

$$\Omega_0 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) \;\middle|\; |a - d| \leq |c| \text{ and } (a - d, c) \neq (0, 0) \right\} .$$

Taking the adjoint by $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for $|\lambda| \geq 2$, we get that $m(\Omega_1) = 0$ for

$$\Omega_1 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) \;\middle|\; |a - d| \leq 4|c| \text{ and } (a - d, c) \neq (0, 0) \right\} .$$

For the same reason, we get that $m(\Omega_1') = 0$ for

$$\Omega_1' := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) \;\middle|\; |a - d| \leq 4|b| \text{ and } (a - d, b) \neq (0, 0) \right\} .$$

Write $X = M_2(F) \setminus F\mathbb{I}$. The matrices with $(a - d, c) = (0, 0)$ belong to Ω_1' unless they are diagonal. Similarly, the matrices with $(a - d, b) = (0, 0)$ belong to Ω_1 unless they are diagonal. So, we find that $m(\Omega) = 0 = m(\Omega')$ for

$$\Omega = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X \;\middle|\; |a - d| \leq 4|c| \right\} \quad \text{and} \quad \Omega' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X \;\middle|\; |a - d| \leq 4|b| \right\} .$$

Put $\Omega'' := g_1 \Omega g_1^{-1}$, so that $m(\Omega'') = 0$. To conclude the proof in the case $k = 2$, it suffices to show that $\Omega \cup \Omega' \cup \Omega'' = X$.

Take $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X \setminus (\Omega \cup \Omega')$. So, $\frac{1}{4}|a - d| > |b|, |c|$. We claim that

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := g_1^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} g_1 = \begin{pmatrix} a - c & a + b - c - d \\ c & c + d \end{pmatrix}$$

belongs to Ω. Since

$$|a' - d'| = |a - d - 2c| \leq |a - d| + 2|c| < \frac{3}{2}|a - d| \quad \text{and} \quad |b'| \geq |a - d| - |c| - |b| > \frac{1}{2}|a - d| ,$$

we indeed get that $|a' - d'| < 3|b'|$. The claim follows and the lemma is proved in the case $k = 2$.

For an arbitrary $k \geq 2$ and fixed $1 \leq p < q \leq k$, the map

$$M_k(F) \to M_2(F) : (x_{ij}) \mapsto \begin{pmatrix} x_{pp} & x_{pq} \\ x_{qp} & x_{qq} \end{pmatrix}$$

is $\operatorname{Ad} \mathrm{SL}(2, F)$-equivariant. So, an $\operatorname{Ad} \mathrm{SL}(k, F)$-invariant mean m on $M_k(F)$ is supported on $\{(x_{ij}) \in M_k(F) \mid x_{pp} = x_{qq}, x_{pq} = x_{qp} = 0\}$. Since $F\mathbb{I}$ is the intersection of these sets, m is supported on $F\mathbb{I}$. □

5 Weak amenability of rigid C*-tensor categories

Following [PV14, Definition 5.1], a rigid C*-tensor category is called *weakly amenable* if there exists a sequence of completely bounded (cb) multipliers $\varphi_n : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ (see Definition 3.2) converging to 1 pointwise, with $\limsup_n \|\varphi_n\|_{\mathrm{cb}} < \infty$ and with φ_n finitely supported for every n.

Recall from the first paragraphs of Section 3 the definition of the tube $*$-algebra $\mathcal{A}$ of $\mathcal{C}$ with respect to a full family of objects in $\mathcal{C}$. To every function $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$, we associate the linear map

$$\theta_\varphi : \mathcal{A} \to \mathcal{A} : \theta_\varphi(V) = \varphi(\alpha)\, V \quad \text{for all} \;\; V \in (i\alpha, \alpha j) \; .$$

We define $\|\theta_\varphi\|_{\mathrm{cb}}$ by viewing $\mathcal{A}$ inside its reduced C*-algebra, i.e. by viewing $\mathcal{A} \subset B(L^2(\mathcal{A}, \mathrm{Tr}))$, where Tr is the canonical trace on $\mathcal{A}$. We also consider the von Neumann algebra $\mathcal{A}''$ generated by $\mathcal{A}$ acting on $L^2(\mathcal{A}, \mathrm{Tr})$.

In the following result, we clarify the link between the complete boundedness of φ in the sense of Definition 3.2 and the complete boundedness of the map θ_φ.

Proposition 5.1. *Let $\mathcal{C}$ be a rigid C*-tensor category. Denote by $\mathcal{A}$ the tube $*$-algebra of $\mathcal{C}$ with respect to a full family of objects in $\mathcal{C}$. Let $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ be any function.*

Then, $\|\varphi\|_{\mathrm{cb}} = \|\theta_\varphi\|_{\mathrm{cb}}$. If this cb-norm is finite, we can uniquely extend θ_φ to a normal completely bounded map on $\mathcal{A}''$ having the same cb-norm.

Proof. For any family J of objects, we can define the tube $*$-algebra $\mathcal{A}_J$ and the linear map $\theta^J_\varphi : \mathcal{A}_J \to \mathcal{A}_J$. By strong Morita equivalence, we have $\|\theta^J_\varphi\|_{\mathrm{cb}} = \|\theta_\varphi\|_{\mathrm{cb}}$ whenever J is full and we have $\|\theta^J_\varphi\|_{\mathrm{cb}} \leq \|\theta_\varphi\|_{\mathrm{cb}}$ for arbitrary J. Also, using standard solutions for the conjugate equations, we get natural linear maps $(i\alpha, \alpha j) \to (\overline{j}\overline{\alpha}, \overline{\alpha}\overline{i})$ and they define a trace preserving $*$-anti-isomorphism of $\mathcal{A}_J$ onto $\mathcal{A}_{\overline{J}}$. Defining $\widetilde{\varphi} : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ by $\widetilde{\varphi}(\alpha) = \varphi(\overline{\alpha})$ for all $\alpha \in \mathrm{Irr}(\mathcal{C})$, it follows that $\|\theta_\varphi\|_{\mathrm{cb}} = \|\theta_{\widetilde{\varphi}}\|_{\mathrm{cb}}$ and it follows that θ_φ extends to a normal completely bounded map on $\mathcal{A}''$ if and only if $\theta_{\widetilde{\varphi}}$ extends to $\mathcal{A}''$.

So, it suffices to prove that $\|\varphi\|_{\mathrm{cb}} = \|\theta_{\widetilde{\varphi}}\|_{\mathrm{cb}}$ and that in the case where $\|\varphi\|_{\mathrm{cb}} < \infty$, we can extend $\theta_{\widetilde{\varphi}}$ to a normal completely bounded map on $\mathcal{A}''$. First assume that $\|\theta_{\widetilde{\varphi}}\|_{\mathrm{cb}} \leq \kappa$. Fix arbitrary objects $\alpha, \beta \in \mathcal{C}$ and write $\Psi^\varphi_{\alpha|\beta} := \Psi^\varphi_{\alpha|\beta,\alpha|\beta}$. We prove that $\|\Psi^\varphi_{\alpha|\beta}\|_{\mathrm{cb}} \leq \kappa$. Since α, β were arbitrary, it then follows that $\|\varphi\|_{\mathrm{cb}} \leq \kappa$.

Note that $(\alpha\beta, \alpha\beta)$ is a finite dimensional C*-algebra. Consider the following three bijective linear maps, making use of standard solutions of the conjugate equations.

$$\begin{aligned}
&\eta_1 : \bigoplus_{\pi\in\mathrm{Irr}(\mathcal{C})} \big((\alpha,\alpha\pi)\otimes(\pi\beta,\beta)\big) \to (\alpha\beta,\alpha\beta) : \eta_1(V\otimes W) = (V\otimes 1)(1\otimes W) \;\; , \\
&\eta_2 : \bigoplus_{\pi\in\mathrm{Irr}(\mathcal{C})} \big((\alpha,\alpha\pi)\otimes(\pi\beta,\beta)\big) \to \bigoplus_{\pi\in\mathrm{Irr}(\mathcal{C})} \big((\alpha\overline{\pi},\alpha)\otimes(\beta,\overline{\pi}\beta)\big) : \\
&\qquad\qquad\qquad\qquad\qquad \eta_2(V\otimes W) = (V\otimes 1)(1\otimes s_\pi)\otimes(t_\pi^*\otimes 1)(1\otimes W) \;\; , \\
&\eta_3 : \bigoplus_{\pi\in\mathrm{Irr}(\mathcal{C})} \big((\alpha\overline{\pi},\alpha)\otimes(\beta,\overline{\pi}\beta)\big) \to \mathcal{A}_{\beta\alpha} : \eta_3(V\otimes W) = (1\otimes V)(W\otimes 1) \;\; .
\end{aligned}$$

A direct computation shows that $\eta := \eta_3\circ\eta_2\circ\eta_1^{-1}$ is a unital faithful $*$-homomorphism of $(\alpha\beta,\alpha\beta)$ to the tube $*$-algebra $\mathcal{A}_{\beta\alpha}$. One also checks that $\theta^{\beta\alpha}_{\widetilde{\varphi}}\circ\eta = \eta\circ\Psi^{\varphi}_{\alpha|\beta}$. So, we get that

$$\|\Psi^{\varphi}_{\alpha|\beta}\|_{\mathrm{cb}} \leq \|\theta^{\beta\alpha}_{\widetilde{\varphi}}\|_{\mathrm{cb}} \leq \|\theta_{\widetilde{\varphi}}\|_{\mathrm{cb}} \leq \kappa \; .$$

Conversely, assume that $\|\varphi\|_{\mathrm{cb}} \leq \kappa$. Define the ind-objects ρ_1 and ρ_2 for $\mathcal{C}$ given by

$$\rho_1 = \bigoplus_{\alpha,i\in\mathrm{Irr}(\mathcal{C})} \alpha i \quad\text{and}\quad \rho_2 = \bigoplus_{\alpha\in\mathrm{Irr}(\mathcal{C})} \alpha \; .$$

Define the type I von Neumann algebra $\mathcal{M}$ of all bounded endomorphisms of $\rho_1\rho_2$. Note that for all $\alpha,i,\beta\in\mathrm{Irr}(\mathcal{C})$, we have the natural projection $p_\alpha\otimes p_i\otimes p_\beta\in\mathcal{M}$ and we have the identification

$$(p_\alpha\otimes p_i\otimes p_\beta)\mathcal{M}(p_\gamma\otimes p_j\otimes p_\delta) = (\alpha i\beta,\gamma j\delta)$$

for all $\alpha,i,\beta,\gamma,j,\delta\in\mathrm{Irr}(\mathcal{C})$. By our assumption, there is a normal completely bounded map $\Psi:\mathcal{M}\to\mathcal{M}$ satisfying

$$\Psi(V) = \Psi^{\varphi}_{\alpha i|\beta,\gamma j|\delta}(V) \quad\text{for all}\;\; V\in(\alpha i\beta,\gamma j\delta) \; .$$

We have $\|\Psi\|_{\mathrm{cb}}\leq\kappa$.

Consider the projection $q\in\mathcal{M}$ given by

$$q = \sum_{\alpha,i\in\mathrm{Irr}(\mathcal{C})} p_{\overline{\alpha}}\otimes p_i\otimes p_\alpha \; .$$

Since $\Psi(qTq) = q\Psi(T)q$ for all $T\in\mathcal{M}$, the map Ψ restricts to a normal completely bounded map on $q\mathcal{M}q$ with $\|\Psi|_{q\mathcal{M}q}\|_{\mathrm{cb}}\leq\kappa$.

Denote by $\mathcal{A}$ the tube $*$-algebra associated with $\mathrm{Irr}(\mathcal{C})$ itself as a full family of objects. We construct a faithful normal $*$-homomorphism $\Theta:\mathcal{A}''\to q\mathcal{M}q$ satisfying $\Psi\circ\Theta = \Theta\circ\theta_{\widetilde{\varphi}}$. Once we have obtained Θ, it follows that $\|\theta_{\widetilde{\varphi}}\|_{\mathrm{cb}}\leq\kappa$ and that $\theta_{\widetilde{\varphi}}$ extends to a normal completely bounded map on $\mathcal{A}''$.

To construct Θ, define the Hilbert space

$$\mathcal{H} = \bigoplus_{\alpha,i,j\in\mathrm{Irr}(\mathcal{C})} (\overline{\alpha}i\alpha,j)$$

and observe that we have the natural faithful normal $*$-homomorphism $\pi:q\mathcal{M}q\to B(\mathcal{H})$ given by left multiplication. Also consider the unitary operator

$$U : L^2(\mathcal{A},\mathrm{Tr})\to\mathcal{H} : U(V) = d(\alpha)^{-1/2}\,(1\otimes V)(t_\alpha\otimes 1) \quad\text{for all}\;\; V\in(i\alpha,\alpha j) \; .$$

We claim that Θ can be constructed such that $\pi(\Theta(V)) = UVU^*$ for all $V \in \mathcal{A}$. To prove this claim, fix $i, \alpha, j \in \mathrm{Irr}(\mathcal{C})$ and $V \in (i\alpha, \alpha j)$. For all $\gamma, \beta \in \mathrm{Irr}(\mathcal{C})$, define the element $W_{\gamma,\beta} \in (\overline{\gamma} i \gamma, \overline{\beta} j \beta)$ given by the finite sum

$$W_{\gamma,\beta} = \sum_{Z \in \mathrm{onb}(\overline{\gamma}\alpha, \overline{\beta})} d(\beta)^{1/2}\, d(\gamma)^{1/2}\, (1 \otimes 1 \otimes \widetilde{Z})\, (1 \otimes V \otimes 1)\, (Z \otimes 1 \otimes 1)\;, \tag{5.1}$$

where $\widetilde{Z} = (1 \otimes t_\beta^*)(1 \otimes Z^* \otimes 1)(s_\gamma \otimes 1)$ belongs to $(\gamma, \alpha\beta)$. A direct computation shows that

$$\langle \pi(W_{\gamma,\beta})\, U(X), U(Y)\rangle = \langle V \cdot X, Y\rangle$$

for all $X \in (j\beta, \beta k)$ and $Y \in (i\gamma, \gamma l)$. So, there is a unique element $\Theta(V) \in (1 \otimes p_i \otimes 1) q\mathcal{M}q (1 \otimes p_j \otimes 1)$ satisfying

$$(p_{\overline{\gamma}} \otimes p_i \otimes p_\gamma)\, \Theta(V)\, (p_{\overline{\beta}} \otimes p_j \otimes p_\beta) = W_{\gamma,\beta}$$

for all $\gamma, \beta \in \mathrm{Irr}(\mathcal{C})$ and $\pi(\Theta(V)) = UVU^*$.

We have defined a faithful normal $*$-homomorphism $\Theta : \mathcal{A}'' \to q\mathcal{M}q$. It remains to prove that $\Psi \circ \Theta = \Theta \circ \theta_{\widetilde{\varphi}}$. Using (5.1), it suffices to prove that

$$\Psi^{\varphi}_{\overline{\gamma} i | \alpha\beta, \overline{\gamma}\alpha j | \beta}(1 \otimes V \otimes 1) = \varphi(\overline{\alpha})\, 1 \otimes V \otimes 1\;. \tag{5.2}$$

The left hand side of (5.2) equals $1 \otimes \Psi^{\varphi}_{i|\alpha, \alpha j|\varepsilon}(V) \otimes 1$. Writing $V = (T \otimes 1)(1 \otimes 1 \otimes s_{\overline{\alpha}})$ with $T \in (i, \alpha j \overline{\alpha})$, we have

$$\begin{aligned}\Psi^{\varphi}_{i|\alpha, \alpha j|\varepsilon}(V) &= (T \otimes 1)\, \Psi^{\varphi}_{\alpha j \overline{\alpha}|\alpha, \alpha j|\varepsilon}(1 \otimes 1 \otimes s_{\overline{\alpha}}) = (T \otimes 1)(1 \otimes 1 \otimes \Psi^{\varphi}_{\overline{\alpha}|\alpha, \varepsilon|\varepsilon}(s_{\overline{\alpha}})\\ &= \varphi(\overline{\alpha})\, (T \otimes 1)(1 \otimes 1 \otimes s_{\overline{\alpha}}) = \varphi(\overline{\alpha})\, V\;.\end{aligned}$$

So (5.2) holds and the proposition is proven. ☐

6 Weak amenability of $\mathcal{C}(K < G)$

Theorem 6.1. *Let G be a totally disconnected group and $K < G$ a compact open subgroup. Denote by $\mathcal{C}$ the rigid C*-tensor category of finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules.*

Then $\mathcal{C}$ is weakly amenable if and only if G is weakly amenable and there exists a sequence of probability measures $\omega_n \in C_0(G)^*$ that are absolutely continuous with respect to the Haar measure and such that $\omega_n \to \delta_e$ weakly and $\|\omega_n \circ \mathrm{Ad}\, x - \omega_n\| \to 0$ uniformly on compact sets of $x \in G$.*

In that case, the Cowling-Haagerup constant $\Lambda(\mathcal{C})$ of $\mathcal{C}$ equals $\Lambda(G)$.

In order to prove Theorem 6.1, we must describe the cb-multipliers on $\mathcal{C}$ in terms of completely bounded multipliers on the C*-algebra $C_0(G) \rtimes^r_{\mathrm{Ad}} G$.

We denote by $\mathrm{Pol}(G)$ the $*$-algebra of locally constant, compactly supported functions on G. Note that $\mathrm{Pol}(G)$ is the linear span of the functions of the form 1_{Ly} where $y \in G$ and $L < G$ is a compact open subgroup. Also note that for any compact open subgroup $K_0 < G$, $\mathrm{Pol}(K_0)$ coincides with the $*$-algebra of coefficients of finite dimensional unitary representations of K_0. We define $\mathcal{E}(G) = \mathrm{Pol}(G)^*$ as the space of all linear maps from $\mathrm{Pol}(G)$ to $\mathbb{C}$. Note that $\mathcal{E}(G)$ can be identified with the space of finitely additive, complex measures on the space $\mathcal{F}(G)$ of compact open subsets of G.

When $K_0 < G$ is a compact open subgroup, we say that a map $\mu : G \to \mathcal{E}(G)$ is K_0-equivariant if

$$\mu(kxk') = \mu(x) \circ \mathrm{Ad}\, k^{-1} \quad \text{for all} \;\; k, k' \in K_0 \; .$$

Note that this implies that $\mu(x)$ is $\mathrm{Ad}(K_0 \cap xK_0x^{-1})$-invariant for all $x \in G$.

As in (3.4), we associate to every $x \in G$ and $\pi \in \mathrm{Irr}(K \cap xKx^{-1})$ the irreducible object $(\pi, x) \in \mathrm{Irr}(\mathcal{C})$ defined as the irreducible G-$L^\infty(G/K)$-$L^\infty(G/K)$-module $\mathcal{H}$ such that π is isomorphic with the representation of $K \cap xKx^{-1}$ on $1_{xK} \cdot \mathcal{H} \cdot 1_{eK}$. The formula

$$\varphi(\pi, x) = \dim(\pi)^{-1}\, \mu(x)(\chi_\pi) \tag{6.1}$$

then gives a bijection between the set of all functions $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ and the set of all K-equivariant maps $\mu : G \to \mathcal{E}(G)$ with the property that $\mu(x)$ is supported on $K \cap xKx^{-1}$ for every $x \in G$.

Denote by $\mathcal{P} = \mathrm{Pol}(L^\infty(G) \rtimes_{\mathrm{Ad}} G)$ the dense $*$-subalgebra defined in (3.3). We always equip $\mathcal{P}$ with the operator space structure inherited from $\mathcal{P} \subset L^\infty(G) \rtimes_{\mathrm{Ad}} G$. As in Section 5, to every function $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ is associated a linear map $\theta_\varphi : \mathcal{A} \to \mathcal{A}$ on the tube $*$-algebra $\mathcal{A}$ of $\mathcal{C}$. We now explain how to associate to any K_0-equivariant map $\mu : G \to \mathcal{E}(G)$ a linear map $\Psi_\mu : \mathcal{P} \to \mathcal{P}$. When φ and μ are related by (6.1) and $\Theta : \mathcal{A} \to \mathcal{P}$ is the $*$-anti-isomorphism of Theorem 3.1, it will turn out that $\Psi_\mu \circ \Theta = \Theta \circ \theta_\varphi$, so that in particular, $\|\theta_\varphi\|_{\mathrm{cb}} = \|\Psi_\mu\|_{\mathrm{cb}}$. We will further prove a criterion for Ψ_μ to be completely bounded and that will be the main tool to prove Theorem 6.1.

Denote $\Delta : L^\infty(G) \to L^\infty(G \times G) : \Delta(F)(g,h) = F(gh)$. For every $\mu \in \mathcal{E}(G)$, the linear map

$$\psi_\mu : \mathrm{Pol}(G) \to \mathrm{Pol}(G) : \psi_\mu(F) = (\mu \otimes \mathrm{id})\Delta(F)$$

is well defined. When $\mu : G \to \mathcal{E}(G)$ is K_0-equivariant with respect to the compact open subgroup $K_0 < G$, we define

$$\Psi_\mu : \mathcal{P} \to \mathcal{P} : \Psi_\mu(F\, u_x\, p_L) = \psi_{\mu(x)}(F)\, u_x\, p_L$$

for every $F \in \mathrm{Pol}(G)$, $x \in G$ and open subgroup $L < K_0$.

Lemma 6.2. *Denote by $\Theta : \mathcal{A} \to \mathcal{P}$ the $*$-anti-isomorphism constructed in Theorem 3.1 between the tube $*$-algebra $\mathcal{A}$ and $\mathcal{P} = \mathrm{Pol}(L^\infty(G) \rtimes_{\mathrm{Ad}} G)$. Let $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ be any function and denote by $\mu : G \to \mathcal{E}(G)$ the associated K-equivariant map given by (6.1) with $\mu(x)$ supported in $K \cap xKx^{-1}$ for all $x \in G$. Then, $\Psi_\mu \circ \Theta = \Theta \circ \theta_\varphi$.*

Proof. The result follows from a direct computation using (3.7). □

We prove the following technical result in exactly the same way as [J91].

Lemma 6.3. *Let $K_0, K < G$ be compact open subgroups and $\mu : G \to \mathcal{E}(G)$ a K_0-equivariant map. Let $\kappa \geq 0$. Then the following conditions are equivalent.*

1. *Ψ_μ extends to a completely bounded map on $C_0(G) \rtimes^r_{\mathrm{Ad}} G$ with $\|\Psi_\mu\|_{\mathrm{cb}} \leq \kappa$.*
2. *Ψ_μ extends to a normal completely bounded map on $L^\infty(G) \rtimes_{\mathrm{Ad}} G$ with $\|\Psi_\mu\|_{\mathrm{cb}} \leq \kappa$.*
3. *There exists a nondegenerate $*$-representation $\pi : C_0(G) \rtimes^f_{\mathrm{Ad}} G \to B(\mathcal{K})$ and bounded maps $V, W : G \to \mathcal{K}$ such that*

- $V(kxk') = \pi(k)V(x)$ *and* $W(kxk') = \pi(k)W(x)$ *for all* $x \in G$*,* $k \in K_0$ *and* $k' \in K$*,*
- $\mu(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1})V(y), W(z)\rangle$ *for all* $F \in \mathrm{Pol}(G)$ *and* $y, z \in G$*,*
- $\|V\|_\infty \|W\|_\infty \leq \kappa$*.*

In particular, every $\mu(x)$ *is an actual complex measure on* G*, i.e.* $\mu(x) \in C_0(G)^*$*.*

Proof. 1 $\Rightarrow$ 3. Denote $P = C_0(G) \rtimes^r_{\mathrm{Ad}} G$ and consider the (unique) completely bounded extension of Ψ_μ to P, which we still denote as Ψ_μ. Define the nondegenerate $*$-representation

$$\zeta : P \to B(L^2(G)) : \zeta(F) = F(e)1 \quad \text{and} \quad \zeta(u_x) = \lambda_x$$

for all $F \in C_0(G)$, $x \in G$. Then $\zeta \circ \Psi_\mu : P \to B(L^2(G))$ has cb norm bounded by κ and satisfies

$$(\zeta \circ \Psi_\mu)(u_k S u_{k'}) = \lambda_k \, (\zeta \circ \Psi_\mu)(S) \, \lambda_{k'}$$

for all $S \in P$, $k, k' \in K_0$. By the Stinespring dilation theorem proved in [BO08, Theorem B.7], we can choose a nondegenerate $*$-representation $\pi : P \to B(\mathcal{K})$ and bounded operators $\mathcal{V}, \mathcal{W} : L^2(G) \to \mathcal{K}$ such that

- $(\zeta \circ \Psi_\mu)(S) = \mathcal{W}^* \pi(S) \mathcal{V}$ for all $S \in P$,
- $\mathcal{V}\lambda_k = \pi(k)\mathcal{V}$ and $\mathcal{W}\lambda_k = \pi(k)\mathcal{W}$ for all $k \in K_0$,
- $\|\mathcal{V}\| \, \|\mathcal{W}\| = \|\Psi_\mu\|_{\mathrm{cb}} \leq \kappa$.

We normalize the left Haar measure on G such that $\lambda(K) = 1$ and define the maps $V, W : G \to \mathcal{K}$ given by $V(y) = \mathcal{V}(1_{yK})$ and $W(z) = \mathcal{W}(1_{zK})$. By construction, 3 holds.

3 $\Rightarrow$ 2. Write $P'' = L^\infty(G) \rtimes_{\mathrm{Ad}} G$. Denote by $\pi_r : P'' \to B(L^2(G \times G))$ the standard representation given by

$$(\pi_r(F)\xi)(g, h) = F(hgh^{-1})\xi(g, h) \quad \text{and} \quad (\pi_r(u_x)\xi)(g, h) = \xi(g, x^{-1}h)$$

for all $g, h, x \in G$, $F \in L^\infty(G)$. For every nondegenerate $*$-representation $\pi : C_0(G) \rtimes^f_{\mathrm{Ad}} G \to B(\mathcal{K})$, there is a unique normal $*$-homomorphism $\widetilde{\pi} : P'' \to B(\mathcal{K} \otimes L^2(G \times G))$ satisfying

$$\widetilde{\pi}(F) = (\pi \otimes \pi_r)\Delta(F) \quad \text{and} \quad \widetilde{\pi}(u_x) = \pi(x) \otimes \pi_r(x)$$

for all $F \in C_0(G)$, $x \in G$. Given V and W as in 3, we then define the bounded operators $\mathcal{V}, \mathcal{W} : L^2(G \times G) \to \mathcal{K} \otimes L^2(G \times G)$ by

$$(\mathcal{V}\xi)(g, h) = \xi(g, h)V(h) \quad \text{and} \quad (\mathcal{W}\xi)(g, h) = \xi(g, h)W(h)$$

for all $g, h \in G$. Note that $\|\mathcal{V}\| = \|V\|_\infty$ and $\|\mathcal{W}\| = \|W\|_\infty$. Since $\Psi_\mu(T) = \mathcal{W}^* \widetilde{\pi}(T)\mathcal{V}$ for all $T \in \mathcal{P}$, it follows that 2 holds.

2 $\Rightarrow$ 1 is trivial. $\square$

We are now ready to prove Theorem 6.1. We follow closely the proof of [O10, Theorem A].

Proof of Theorem 6.1. We define $\mathcal{Q}(G)$ as the set of all maps $\mu : G \to \mathcal{E}(G)$ satisfying the following properties:

- there exists a compact open subgroup $K_0 < G$ such that μ is K_0-equivariant,

- for every $x \in G$, we have that $\mu(x) \in C_0(G)^*$, $\mu(x)$ is compactly supported and $\mu(x)$ is absolutely continuous with respect to the Haar measure,
- $\|\Psi_\mu\|_{\mathrm{cb}} < \infty$.

Writing $\|\mu\|_{\mathrm{cb}} := \|\Psi_\mu\|_{\mathrm{cb}}$, we call a sequence $\mu_n \in \mathcal{Q}(G)$ a cbai (completely bounded approximate identity) if

- $\limsup_n \|\mu_n\|_{\mathrm{cb}} < \infty$,
- for every $F \in C_0(G)$, we have that $\mu_n(x)(F) \to F(e)$ uniformly on compact sets of $x \in G$,
- for every n, we have that μ_n has compact support (i.e. $\mu_n(x) = 0$ for all x outside a compact subset of G).

If a cbai exists, we define $\Gamma(G)$ as the smallest possible value of $\limsup_n \|\mu_n\|_{\mathrm{cb}}$, where (μ_n) runs over all cbai. Note that this smallest possible value is always attained by a cbai.

First assume that $\mathcal{C}$ is weakly amenable. By Proposition 5.1, we can take a sequence of finitely supported functions $\varphi_n : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ converging to 1 pointwise and satisfying $\limsup_n \|\theta_{\varphi_n}\|_{\mathrm{cb}} = \Lambda(\mathcal{C})$ where $\theta_{\varphi_n} : \mathcal{A} \to \mathcal{A}$ as before. Define the K-equivariant maps $\mu_n : G \to \mathcal{E}(G)$ associated with φ_n by (6.1).

For a fixed n and a fixed $x \in G$, there are only finitely many $\pi \in \mathrm{Irr}(K \cap xKx^{-1})$ such that $\varphi_n(\pi, x) \neq 0$. So, $\mu_n(x)$ is an actual complex measure on $K \cap xKx^{-1}$ that is absolutely continuous with respect to the Haar measure (and with the Radon-Nikodym derivative being in $\mathrm{Pol}(K \cap xKx^{-1})$). By Lemma 6.2, $\|\Psi_{\mu_n}\|_{\mathrm{cb}} = \|\theta_{\varphi_n}\|_{\mathrm{cb}} < \infty$. So, $\mu_n \in \mathcal{Q}(G)$ and the sequence (μ_n) is a cbai with $\limsup_n \|\mu_n\|_{\mathrm{cb}} \leq \Lambda(\mathcal{C})$. Thus, $\Gamma(G) \leq \Lambda(\mathcal{C})$. Write $\kappa = \Gamma(G)^{1/2}$.

For every map $\mu : G \to \mathcal{E}(G)$, we define

$$\overline{\mu} : G \to \mathcal{E}(G) : \overline{\mu}(x)(F) = \overline{(\mu(x^{-1}) \circ \mathrm{Ad}(x^{-1}))(\overline{F})} \; .$$

If μ is K_0-equivariant, also $\overline{\mu}$ is K_0-equivariant and $\Psi_{\overline{\mu}}(T) = (\Psi_\mu(T^*))^*$ for all $T \in \mathcal{P}$. So, $\|\overline{\mu}\|_{\mathrm{cb}} = \|\mu\|_{\mathrm{cb}}$. Also, if (μ_n) is a cbai, then $(\overline{\mu_n})$ is a cbai.

Since $\Gamma(G) = \kappa^2 < \infty$ and using Lemma 6.3, we can take a cbai (μ_n), a nondegenerate $*$-representation $\pi : C_0(G) \rtimes^f_{\mathrm{Ad}} G \to B(\mathcal{K})$ and bounded functions $V_n, W_n : G \to \mathcal{K}$ as in Lemma 6.3.3 with

$$\lim_n \|V_n\|_\infty = \kappa = \lim_n \|W_n\|_\infty \; .$$

Replacing μ_n by $(\mu_n + \overline{\mu_n})/2$, we may assume that $\mu_n = \overline{\mu_n}$ for all n. It then follows that both formulas

$$\mu_n(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1})V_n(y), W_n(z) \rangle \quad \text{and}$$
$$\mu_n(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1})W_n(y), V_n(z) \rangle$$

hold for all $F \in C_0(G)$ and $y, z \in G$.

Put $\eta_n := \mu_n(e)$. We prove that $\|\eta_n \circ \mathrm{Ad}\, x - \eta_n\| \to 0$ uniformly on compact sets of $x \in G$. To prove this statement, fix an arbitrary compact subset $C \subset G$ and an arbitrary sequence $x_n \in C$. Define

$$\zeta_n : G \to \mathcal{E}(G) : \zeta_n(x) = \mu_n(x_n x) \circ \mathrm{Ad}\, x_n \; .$$

Since $\Psi_{\zeta_n}(T) = u^*_{x_n} \Psi_{\mu_n}(u_{x_n} T)$, it follows that (ζ_n) is a cbai. Also note that for all $y, z \in G$ and $F \in C_0(G)$, we have

$$\zeta_n(zy^{-1})(F) = \langle \pi((\mathrm{Ad}\, x_n)(F))\, \pi(x_n zy^{-1})\, V_n(y), W_n(x_n z) \rangle = \langle \pi(F)\pi(zy^{-1})V_n(y), W'_n(z) \rangle \; ,$$

with $W_n'(z) = \pi(x_n)^* W_n(x_n z)$. Then also $(\mu_n + \zeta_n)/2$ is a cbai satisfying

$$\frac{1}{2}(\mu_n + \zeta_n)(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1})V_n(y), (W_n(z) + W_n'(z))/2 \rangle$$

for all $y, z \in G$ and $F \in C_0(G)$. We conclude that

$$\begin{aligned}\kappa^2 &\leq \liminf_n \|V_n\|_\infty \, \|(W_n + W_n')/2\|_\infty = \kappa \, \liminf_n \|(W_n + W_n')/2\|_\infty \\ &\leq \kappa \, \limsup_n \|(W_n + W_n')/2\|_\infty \leq \kappa \, \frac{1}{2} \limsup_n \bigl(\|W_n\|_\infty + \|W_n'\|_\infty\bigr) = \kappa^2 \, .\end{aligned}$$

Therefore, $\lim_n \|(W_n + W_n')/2\|_\infty = \kappa$. So, we can choose $z_n \in G$ such that $\lim_n \|(W_n(z_n) + W_n'(z_n))/2\| = \kappa$. Since also $\limsup_n \|W_n(z_n)\| \leq \kappa$ and $\limsup_n \|W_n'(z_n)\| \leq \kappa$, the parallelogram law implies that $\lim_n \|W_n(z_n) - W_n'(z_n)\| = 0$.

Since for all $F \in C_0(G)$,

$$\begin{aligned}&\zeta_n(e)(F) = \zeta_n(z_n z_n^{-1})(F) = \langle \pi(F)V_n(z_n), W_n'(z_n)\rangle \quad \text{and} \\ &\mu_n(e)(F) = \mu_n(z_n z_n^{-1})(F) = \langle \pi(F)V_n(z_n), W_n(z_n)\rangle \, ,\end{aligned}$$

it follows that $\lim_n \|\zeta_n(e) - \mu_n(e)\| = 0$. This means that $\lim_n \|\mu_n(x_n) \circ \operatorname{Ad} x_n - \mu_n(e)\| = 0$. Since the sequence $x_n \in C$ was arbitrary, we have proved that $\lim_n \|\mu_n(x) - \mu_n(e) \circ \operatorname{Ad} x^{-1}\| = 0$ uniformly on compact sets of $x \in G$.

Reasoning in a similar way with $\zeta_n : G \to \mathcal{E}(G) : \zeta_n(x) = \mu_n(x x_n^{-1})$, which satisfies

$$\zeta_n(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1})V_n'(y), W_n(z)\rangle$$

with $V_n'(y) = \pi(x_n)^* V_n(x_n y)$, we also find that $\lim_n \|\mu_n(x) - \mu_n(e)\| = 0$ uniformly on compact sets of $x \in G$. Both statements together imply that $\|\eta_n \circ \operatorname{Ad} x - \eta_n\| \to 0$ uniformly on compact sets of $x \in G$.

We next claim that for every $H \in \operatorname{Pol}(G)$ with $H(e) = 1$ and $\|H\|_\infty = 1$, we have that $\lim_n \|\eta_n \cdot H - \eta_n\| = 0$. To prove this claim, define

$$\zeta_n : G \to \mathcal{E}(G) : \zeta_n(x)(F) = \mu_n(x)(HF) \, .$$

Since $\zeta_n(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1})V_n(y), W_n'(z)\rangle$ with $W_n'(z) = \pi(H)^* W_n(z)$ and because the function $H \in \operatorname{Pol}(G)$ is both left and right K_0-invariant for a small enough compact open subgroup $K_0 < G$, it follows from Lemma 6.3 that

$$\|\zeta_n\|_{\mathrm{cb}} \leq \|V_n\|_\infty \, \|W_n'\|_\infty \leq \|V_n\|_\infty \, \|W_n\|_\infty = \|\mu_n\|_{\mathrm{cb}} \, .$$

So again, (ζ_n) and $(\mu_n + \zeta_n)/2$ are cbai. The same reasoning as above gives us a sequence $z_n \in G$ with $\lim_n \|W_n(z_n) - W_n'(z_n)\| = 0$, which allows us to conclude that $\lim_n \|\mu_n(e) - \zeta_n(e)\| = 0$, thus proving the claim.

Altogether, we have proved that $\eta_n \in C_0(G)^*$ is a sequence of complex measures that are absolutely continuous with respect to the Haar measure and that satisfy

- $\|\eta_n - \eta_n \circ \operatorname{Ad} x\| \to 0$ uniformly on compact sets of $x \in G$,
- $\|\eta_n \cdot 1_L - \eta_n\| \to 0$ for every compact open subset $L \subset G$ with $e \in L$,
- $\eta_n(F) \to F(e)$ for every $F \in C_0(G)$.

In particular, $\liminf_n \|\eta_n\| \geq 1$. But then $\omega_n := \|\eta_n\|^{-1} |\eta_n|$ is a sequence of probability measures on G that are absolutely continuous with respect to the Haar measure and satisfy $\omega_n \to \delta_e$ weakly* and $\|\omega_n \circ \operatorname{Ad} x - \omega_n\| \to 0$ uniformly on compact sets of $x \in G$.

By Lemma 6.3, the maps Ψ_{μ_n} extend to normal cb maps on $L^\infty(G) \rtimes_{\mathrm{Ad}} G$. Restricting to $L(G)$, we obtain the compactly supported Herz-Schur multipliers

$$L(G) \to L(G) : u_x \mapsto \gamma_n(x) u_x \quad \text{for all } \ x \in G \; ,$$

where $\gamma_n : G \to \mathbb{C}$ is the compactly supported, locally constant function given by $\gamma_n(x) = \mu_n(x)(1)$. So, G is weakly amenable and

$$\Lambda(G) \leq \limsup_n \|\Psi_{\mu_n}|_{L(G)}\|_{\mathrm{cb}} \leq \limsup_n \|\Psi_{\mu_n}\| \leq \Lambda(\mathcal{C}) \; .$$

Conversely, assume that G is weakly amenable and that there exists a sequence of probability measures $\omega_n \in C_0(G)^*$ that are absolutely continuous with respect to the Haar measure and such that $\omega_n \to \delta_e$ weakly* and $\|\omega_n \circ \operatorname{Ad} x - \omega_n\| \to 0$ uniformly on compact sets of $x \in G$.

Since G is weakly amenable, we can take a sequence of K-biinvariant Herz-Schur multipliers $\zeta_n : G \to \mathbb{C}$ having compact support, converging to 1 uniformly on compacta and satisfying $\limsup_n \|\zeta_n\|_{\mathrm{cb}} = \Lambda(G)$.

Denote by $\operatorname{Pol}(G)^+$ the set of positive, locally constant, compactly supported functions on G. Denote by $h \in C_0(G)^*$ the Haar measure on the compact open subgroup $K < G$. Approximating ω_n, we may assume that $\omega_n = h \cdot \xi_n^2$, where ξ_n is a sequence of $\operatorname{Ad} K$-invariant functions in $\operatorname{Pol}(K)^+$. Define the representation $\pi : C_0(G) \rtimes^f_{\mathrm{Ad}} G \to B(L^2(G))$ given by

$$(\pi(F)\xi)(g) = F(g)\xi(g) \quad \text{and} \quad (\pi(x)\xi)(g) = \Delta(x)^{1/2}\, \xi(x^{-1}gx)$$

for all $F \in C_0(G)$, $\xi \in L^2(G)$ and $x, g \in G$. We then define the K-equivariant map

$$\mu_n : G \to C_0(G)^* : \mu_n(x)(F) = \zeta_n(x) \, \langle \pi(F)\pi(x)\xi_n, \xi_n \rangle \; .$$

Since ξ_n is an $\operatorname{Ad} K$-invariant element of $\operatorname{Pol}(K)$ and $\pi(x)\xi_n$ is an $\operatorname{Ad}(xKx^{-1})$-invariant element of $\operatorname{Pol}(xKx^{-1})$, we get that $\mu_n(x)$ is an $\operatorname{Ad}(K \cap xKx^{-1})$-invariant complex measure supported on $K \cap xKx^{-1}$ and having a density in $\operatorname{Pol}(K \cap xKx^{-1})$ with respect to the Haar measure. Since moreover ζ_n is compactly supported, it follows that the functions $\varphi_n : \operatorname{Irr}(\mathcal{C}) \to \mathbb{C}$ associated with μ_n through (6.1) are finitely supported.

Since $\|\omega_n \circ \operatorname{Ad} x - \omega_n\| \to 0$ for every $x \in G$, we have that $\|\pi(x)\xi_n - \xi_n\| \to 0$ for every $x \in G$. Since $\omega_n \to \delta_e$ weakly*, we have that $\langle \pi(F)\xi_n, \xi_n \rangle \to F(e)$ for every $F \in C_0(G)$. Both together imply that $\varphi_n \to 1$ pointwise.

To conclude the proof of the theorem, by Lemma 6.2, it suffices to prove that $\limsup_n \|\mu_n\|_{\mathrm{cb}} \leq \Lambda(G)$.

Since ζ_n is a K-biinvariant Herz-Schur multiplier on G, we can choose a Hilbert space $\mathcal{K}$ and K-biinvariant functions $V_n, W_n : G \to \mathcal{K}$ such that

$$\|V_n\|_\infty \, \|W_n\|_\infty = \|\zeta_n\|_{\mathrm{cb}} \quad \text{and} \quad \zeta_n(zy^{-1}) = \langle V_n(y), W_n(z) \rangle \tag{6.2}$$

for all $y, z \in G$. We equip $L^2(G) \otimes \mathcal{K}$ with the $*$-representation of $C_0(G) \rtimes^f_{\mathrm{Ad}} G$ given by $\pi(\,\cdot\,) \otimes 1$. We define the bounded maps

$$\mathcal{V}_n : G \to L^2(G) \otimes \mathcal{K} : \mathcal{V}_n(y) = \xi_n \otimes V_n(y) \quad \text{and} \quad \mathcal{W}_n : G \to L^2(G) \otimes \mathcal{K} : \mathcal{W}_n(y) = \xi_n \otimes W_n(y) \; .$$

One checks that

$$\mu_n(zy^{-1})(F) = \langle (\pi(F)\pi(zy^{-1}) \otimes 1)\mathcal{V}_n(y), \mathcal{W}_n(z)\rangle$$

for all y, z and that all other conditions in Lemma 6.3.3 are satisfied, with $\|\mathcal{V}_n\|_\infty = \|V_n\|_\infty$ and $\|\mathcal{W}_n\|_\infty = \|W_n\|_\infty$. So, we conclude that

$$\limsup_n \|\mu_n\|_{\mathrm{cb}} \leq \limsup_n \|\zeta_n\|_{\mathrm{cb}} = \Lambda(G)$$

and this ends the proof of the theorem. □

Example 6.4. Taking G as in Example 4.4, the category $\mathcal{C}$ is weakly amenable with $\Lambda(\mathcal{C}) = 1$. Indeed, G is weakly amenable with $\Lambda(G) = 1$ and the probability measures μ_n constructed in Example 4.4 are absolutely continuous with respect to the Haar measure, so that the result follows from Theorem 6.1.

Taking $G = \mathrm{SL}(2, F)$ as in Proposition 4.2, we get that $\mathcal{C}$ is not weakly amenable, although G is weakly amenable with $\Lambda(G) = 1$.

References

[A14] Y. Arano, Unitary spherical representations of Drinfeld doubles. *J. Reine Angew. Math.*, to appear. arXiv:1410.6238

[BHV08] B. Bekka, P. de la Harpe and A. Valette, Kazhdan's property (T). Cambridge University Press, Cambridge, 2008.

[Br14] A. Brothier, Weak amenability for subfactors. *Internat. J. Math.* **26** (2015), art. id. 1550048.

[BO08] N.P. Brown and N. Ozawa, C*-algebras and finite-dimensional approximations. *Graduate Studies in Mathematics* **88**. American Mathematical Society, Providence, 2008.

[B10] R.D. Burstein, Automorphisms of the bipartite graph planar algebra. *J. Funct. Anal.* **259** (2010), 2384-2403.

[DFY13] K. De Commer, A. Freslon and M. Yamashita, CCAP for universal discrete quantum groups. *Comm. Math. Phys.* **331** (2014), 677-701.

[DV10] S. Deprez and S. Vaes, A classification of all finite index subfactors for a class of group-measure space II_1 factors. *J. Noncommut. Geom.* **5** (2011), 523-545.

[C05] Y. de Cornulier, Finitely presentable, non-Hopfian groups with Kazhdan's property (T) and infinite outer automorphism group. *Proc. Amer. Math. Soc.* **135** (2007), 951-959.

[GJ15] S. Ghosh and C. Jones, Annular representation theory for rigid C*-tensor categories. *J. Funct. Anal.* **270** (2016), 1537-1584.

[J91] P. Jolissaint, A characterization of completely bounded multipliers of Fourier algebras. *Colloq. Math.* **63** (1992), 311-313.

[J98] V.F.R Jones, The planar algebra of a bipartite graph. In *Knots in Hellas '98*, World Scientific, 1999, pp. 94-117.

[J99] V.F.R. Jones, Planar algebras, I. arXiv:math.QA/9909027

[LR96] R. Longo and J.E. Roberts, A theory of dimension. *K-Theory* **11** (1997), 103-159.

[MPS08] S. Morrison, E. Peters and N. Snyder, Skein theory for the D_{2n} planar algebras. *J. Pure Appl. Algebra* **214** (2010), 117-139.

[M01] M. Müger, From subfactors to categories and topology, I. Frobenius algebras in and Morita equivalence of tensor categories. *J. Pure Appl. Algebra* **180** (2003), 81-157.

[NT13] S. Neshveyev and L. Tuset, Compact quantum groups and their representation categories. *Cours Spécialisés* **20**. Société Mathématique de France, Paris, 2013.

[NY15a] S. Neshveyev and M. Yamashita, Drinfeld center and representation theory for monoidal categories. *Preprint.* arXiv:1501.07390

[NY15b] S. Neshveyev and M. Yamashita, A few remarks on the tube algebra of a monoidal category. *Preprint.* arXiv:1511.06332

[O93] A. Ocneanu, Chirality for operator algebras. In *Subfactors (Kyuzeso, 1993)*, World Sci. Publ., River Edge, 1994, pp. 39-63.

[O10] N. Ozawa, Examples of groups which are not weakly amenable. *Kyoto J. Math.* **52** (2012), 333-344.

[P94a] S. Popa, Symmetric enveloping algebras, amenability and AFD properties for subfactors. *Math. Res. Lett.* **1** (1994), 409-425.

[P94b] S. Popa, An axiomatization of the lattice of higher relative commutants of a subfactor. *Invent. Math.* **120** (1995), 427-445.

[P99] S. Popa, Some properties of the symmetric enveloping algebra of a subfactor, with applications to amenability and property T. *Doc. Math.* **4** (1999), 665-744.

[P01] S. Popa, On a class of type II_1 factors with Betti numbers invariants. *Ann. of Math.* **163** (2006), 809-899.

[PSV15] S. Popa, D. Shlyakhtenko and S. Vaes, Cohomology and L^2-Betti numbers for subfactors and quasi-regular inclusions. *Preprint.* arXiv:1511.07329

[PV14] S. Popa and S. Vaes, Representation theory for subfactors, λ-lattices and C*-tensor categories, *Commun. Math. Phys.* **340** (2015), 1239-1280.

[V03] S. Vaes, Strictly outer actions of groups and quantum groups. *J. Reine Angew. Math.* **578** (2005), 147-184.

[W88] A. Wassermann, Ergodic actions of compact groups on operator algebras, I. *Ann. of Math.* **130** (1989), 273-319.

[Z98] Y. Zhu, Hecke algebras and representation ring of Hopf algebras. In *First International Congress of Chinese Mathematicians (Beijing, 1998)*, AMS/IP Stud. Adv. Math. **20**, Amer. Math. Soc., Providence, 2001, pp. 219-227.